Superluminal Speed and Doppler Boosting Expressions for Relativistic Jets

$\beta = \text{true speed} = v/\text{c} = (1 - 1/\Gamma^2)^{1/2}$

$\theta = \text{viewing angle between velocity vector and line of sight} = \arctan\left(\frac{2\beta_\text{a}}{\beta_\text{a}^2+\delta^2-1}\right)$

$\Gamma = \text{Lorentz factor} = (1-\beta^2)^{-1/2} = \frac{(\beta_\text{a}^2+\delta^2+1)}{2\delta} = \frac{\delta^{-1}\pm\cos\theta\sqrt{\delta^{-2}-\sin^2\theta}}{\sin^2\theta}$

$\Gamma = \left[1 - \left(\beta_\text{a}^{-1}\sin\theta + \cos\theta\right)^{-2}\right]^{-1/2}$

$\beta_\text{a} = \text{apparent speed} = \beta\sin\theta/(1-\beta\cos\theta) = (2\Gamma\delta - \delta^2 - 1)^{1/2} = \Gamma\delta\beta\sin\theta$

$\delta = \text{Doppler factor} = [\Gamma(1-\beta\cos\theta)]^{-1} = \left[\Gamma - \sqrt{\Gamma^2-1}\,\cos\theta\right]^{-1} = \left(\frac{2\beta_\text{a}}{\tan\theta} + 1 - \beta_\text{a}^2\right)^{1/2}$

$\delta = 1 \text{ for } \theta = \arccos\sqrt{(\Gamma-1)/(\Gamma+1)}$

α = Jet spectral index, where $S \propto \nu^\alpha$, and ν = observing frequency.

D = Doppler flux boost = δ^p, where $p = 2 - \alpha$ for continuous jet, $p = 3 - \alpha$ for flaring region.

J = Jet/counterjet flux density (S) ratio.

$\beta\cos\theta = (J^{1/p} - 1)/(J^{1/p} + 1)$

$J^{1/p} = (1+\beta\cos\theta)(1-\beta\cos\theta)^{-1} = \beta_\text{a}^2 + \delta^2$

Limit Expressions

$\Gamma_\text{min} = \sqrt{1+\beta_\text{a}^2}$

$\Gamma_\text{min} = (\delta + \delta^{-1})/2$

$\beta_\text{min} = \beta_\text{a}(1+\beta_\text{a}^2)^{-1/2}$

$\beta_\text{a,max} = \sqrt{\Gamma^2-1} = \delta\beta = \Gamma\beta$ at $\beta = \cos\theta$ or $\sin\theta = 1/\Gamma$

$\delta_\text{max} = \Gamma + \sqrt{\Gamma^2-1} = \Gamma(1+\beta) \simeq 2\Gamma$

$\delta_\text{min} = \Gamma^{-1} = \Gamma_\text{max} - \sqrt{\Gamma_\text{max}^2 - \beta_\text{a}^2 - 1}$, where Γ_max is the fastest jet speed in the population.

$\theta_\text{max} = 2\arctan(\beta_\text{a}^{-1})$

$\theta_\text{max} = \arcsin(\delta^{-1})$

At Critical Angle $\theta_\text{crit} = \sin^{-1}(1/\Gamma)$ or $\theta_\text{crit} = \cos^{-1}\beta$

$\beta_a = \sqrt{\Gamma^2-1} = \delta\beta = \Gamma\beta$

$\delta = \Gamma$

$J^{1/p} = \beta_\text{a}^2 + \Gamma^2 = 2\Gamma^2 - 1 = \delta^2(2 - \Gamma^{-2})$

Linear Size [in pc]

$$1 \text{ mas} = 4.848 \times 10^{-3}\ d_L\ (1+z)^{-2} \qquad\qquad [d_L] = \text{Mpc}$$

Apparent Velocity [v/c]

$$\begin{aligned}
\beta_a &= \frac{d_L\ \mu}{63.24107\ (1+z)} \qquad [\mu] = \text{mas/yr}, \quad [d_L] = \text{Mpc} \\
&= \frac{47.405\ \mu}{q_o^2\ (1+z)} \left[q_o z + (q_o - 1)\left(\sqrt{2 q_o z + 1} - 1\right)\right] h^{-1} \qquad q_o \neq 0 \\
&= 94.810\ \mu \left[1 - (1+z)^{-1/2}\right] h^{-1} \qquad q_o = 0.5 \\
&= \frac{23.702\ \mu\ z\ (2+z)\ h^{-1}}{(1+z)} \qquad q_o = 0
\end{aligned}$$

Luminosity [W/Hz]

$$\begin{aligned}
L_{\nu_e} &= \frac{4\pi d_L^2\ f_{\nu_{obs}}}{(1+z)^{\alpha+1}} \\
\log L_{\nu_e} &= 20.07791 + \log\left[\frac{d_L^2\ f_{\nu_{obs}}}{(1+z)^{\alpha+1}}\right] \qquad [d_L] = \text{Mpc}, \quad [f_{\nu_{obs}}] = \text{Jy}, \quad f_\nu \propto \nu^\alpha
\end{aligned}$$