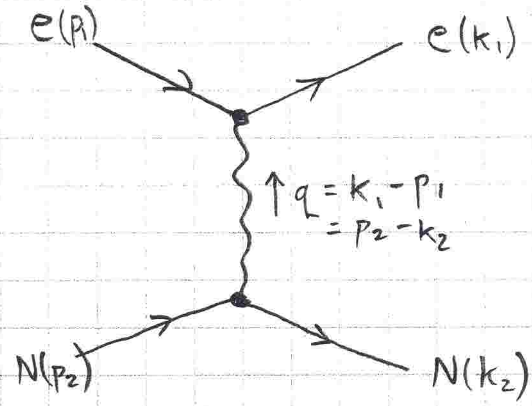
Assignment #6

1. Calculate $|\overline{\mathcal{M}}|^2$ for e^-N scattering in terms of the Mandlestam variables, ignoring the mass of the electron.



$$-i\mathcal{M} = \bar{u}(k_1)(ie\gamma^\mu)u(p_1)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\left(Ze(k_2+p_2)^\nu\right)$$

$$= \frac{Ze^2}{q^2}\,\bar{u}(k_1)\gamma^\mu u(p_1)(k_2+p_2)_\mu$$

$$|\overline{\mathcal{M}}|^2 = \frac{Z^2e^4}{2q^4}\sum_{spins}\left(\bar{u}(k_1)\gamma^\mu u(p_1)\right)\left(\bar{u}(k_1)\gamma^\nu u(p_1)\right)^*(k_2+p_2)_\mu(k_2+p_2)_\nu$$

$$= \frac{Z^2e^4}{2q^4}\mathrm{Tr}\left(u(k_1)\bar{u}(k_1)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\nu\right)(k_2+p_2)_\mu(k_2+p_2)_\nu$$

$$= \frac{Ze^4}{2q^4}\mathrm{Tr}\left(\not{k}_1(\not{k}_2+\not{p}_2)\not{p}_1(\not{k}_2+\not{p}_2)\right) + \mathcal{O}(m_e^2)$$

$$= \frac{Z^2e^4}{2q^4}\Big[4(k_1\cdot k_2 + k_1\cdot p_2)(k_2\cdot p_1 + p_1\cdot p_2) - 2k_1\cdot p_1(k_2\cdot k_2 + 2k_2\cdot p_2 + p_2\cdot p_2)\Big]$$

Next, using
$k_2\cdot k_2 = M^2 = p_2\cdot p_2$
$p_1\cdot p_2 = k_1\cdot k_2 = \frac{1}{2}(s - M^2)$
$k_1\cdot p_2 = k_2\cdot p_1 = \frac{1}{2}(M^2 - u)$
$k_1\cdot p_1 = -t/2 \quad k_2\cdot p_2 = M^2 - t/2$

So $|\overline{m}|^2 = \frac{Z^2 e^4}{t^2}\left(4M^2 t + (s-u)^2 - t^2\right)$

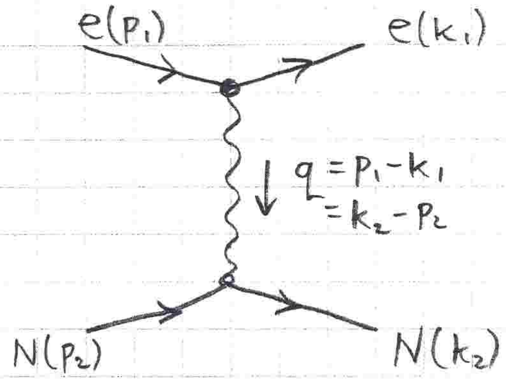
But $u = 2M^2 - s - t$

so this can be written:

$$|\overline{m}|^2 = 4Z^2 e^4\left(\frac{s}{t} + \frac{(s-M^2)^2}{t^2}\right)$$

$$= \frac{4Z^2 e^4}{t^2}\left((s-M^2)^2 + st\right)$$

2.



$$-i\mathcal{M} = \bar{u}(k_1)\left(ie\gamma^\mu\right)u(p_1)\left(-i\frac{g_{\mu\nu}}{q^2}\right)\bar{u}(k_2)\left(iZe\gamma^\nu\right)u(p_2)$$

$$= i\frac{Ze^2}{q^2}\,\bar{u}(k_1)\gamma^\mu u(p_1)\,\bar{u}(k_2)\gamma_\mu u(p_2)$$

$$|\overline{\mathcal{M}}|^2 = \frac{Z^2e^4}{4q^4}\sum_{\text{spins}}\left[\bar{u}(k_1)\gamma^\mu u(p_1)\bar{u}(k_2)\gamma_\mu u(p_2)\right]\left[\bar{u}(k_1)\gamma^\nu u(p_1)\bar{u}(k_2)\gamma_\nu u(p_2)\right]$$

$$= \frac{Z^2e^4}{4q^4}\sum_{\text{spins}}\left[\bar{u}(k_1)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\nu u(k_1)\right]\left[\bar{u}(k_2)\gamma_\mu u(p_2)\bar{u}(p_2)\gamma_\nu u(k_2)\right]$$

$$= \frac{Z^2e^4}{4q^4}\sum_{\text{spins}}\mathrm{Tr}\left[u(k_1)\bar{u}(k_1)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\nu\right]\mathrm{Tr}\left[u(k_2)\bar{u}(k_2)\gamma_\mu u(p_2)\bar{u}(p_2)\gamma_\nu\right]$$

$$= \frac{Z^2e^4}{4q^4}\mathrm{Tr}\left((\not{k}_1+m)\gamma^\mu(\not{p}_1+m)\gamma^\nu\right)\mathrm{Tr}\left((\not{k}_2+M)\gamma_\mu(\not{p}_2+M)\gamma_\nu\right)$$

$$= \frac{8Z^2e^4}{q^4}\left(k_1\cdot k_2\;p_1\cdot p_2 + k_1\cdot p_2\;k_2\cdot p_1 - M^2\,k_1\cdot p_1\right)$$

But $q^2 = t$

$p_1\cdot p_2 = k_1\cdot k_2 = \frac{1}{2}(s - M^2)$

$k_1\cdot p_2 = k_2\cdot p_1 = \frac{1}{2}(M^2 - u)$

$k_1\cdot p_1 = -t/2 \qquad k_2\cdot p_2 = M^2 - t/2$

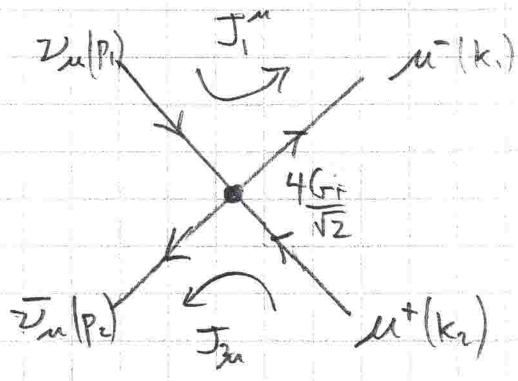
So $$|\overline{\mathcal{M}}|^2 = \frac{2Z^2e^4}{t^2}\left[(s-M^2)^2 + 2M^2 t + (u-M^2)^2 \right]$$

But, using $u = 2M^2 - s - t$, this is

$$|\overline{\mathcal{M}}|^2 = \frac{4Z^2e^4}{t^2}\left[(s-M^2)^2 + t(s+t/2) \right]$$

The only difference compared with the reduced matrix element for the spinless nucleon target is the extra factor of $t^2/2$ in the brackets.

3.



where $J_1^\mu = \bar{u}(k_1)\gamma^\mu \cdot \frac{1}{2}(1-\gamma^5)\, u(p_1)$

$J_{2\mu} = \bar{v}(p_2)\gamma_\mu \frac{1}{2}(1-\gamma^5)\, v(k_2)$

$$-i\mathcal{M} = \frac{4G_F}{\sqrt{2}} J_1^\mu J_{2\mu}$$

$$= \frac{G_F}{\sqrt{2}}\, \bar{u}(k_1)\gamma^\mu(1-\gamma^5)u(p_1)\, \bar{v}(p_2)\gamma_\mu(1-\gamma^5)v(k_2)$$

$$|\mathcal{M}|^2 = \frac{G_F^2}{2}\left[\bar{u}(k_1)\gamma^\mu(1-\gamma^5)u(p_1)\right]\left[\bar{u}(p_1)(1+\gamma^5)\gamma^\nu u(k_1)\right]$$

$$\times \left[\bar{v}(p_2)\gamma_\mu(1-\gamma^5)v(k_2)\right]\left[\bar{v}(k_2)(1+\gamma^5)\gamma_\nu v(p_2)\right]$$

$$= \frac{G_F^2}{2} \mathrm{Tr}\left(u(k_1)\bar{u}(k_1)\gamma^\mu(1-\gamma^5)u(p_1)\bar{u}(p_1)(1+\gamma^5)\gamma^\nu\right)$$

$$\times \mathrm{Tr}\left(v(p_2)\bar{v}(p_2)\gamma_\mu(1-\gamma^5)v(k_2)\bar{v}(k_2)(1+\gamma^5)\gamma_\nu\right)$$

$$\overline{|\mathcal{M}|^2} = \frac{G_F^2}{2} \mathrm{Tr}\left(\not{k}_1\gamma^\mu(1-\gamma^5)\not{p}_1(1+\gamma^5)\gamma^\nu\right) \mathrm{Tr}\left(\not{p}_2\gamma_\mu(1-\gamma^5)\not{k}_2(1+\gamma^5)\gamma_\nu\right)$$

But $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$
and $(1-\gamma^5)^2 = 2(1-\gamma^5)$

$$\overline{|\mathcal{M}|^2} = 2G_F^2\, \mathrm{Tr}\left((1-\gamma^5)\not{k}_1\gamma^\mu\not{p}_1\gamma^\nu\right) \mathrm{Tr}\left((1-\gamma^5)\not{p}_2\gamma_\mu\not{k}_2\gamma_\nu\right)$$

$$= 128\, G_F^2 \cdot (k_1\cdot p_2)(k_2\cdot p_1) = 32\, G_F^2\, u^2$$

But $\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{s} \frac{|\vec{k}|}{|\vec{p}|} |\overline{\mathcal{M}}|^2$

$= \frac{G_F^2}{2\pi^2} \cdot \frac{u^2}{s}$

But $u = -2k_1 \cdot p_2 = -2(E_1E_2' + E_1E_2' \cos\theta)$

$= -\frac{s}{2}(1 + \cos\theta)$

So $\frac{d\sigma}{d\Omega} = \frac{G_F^2}{8\pi^2} s(1 + \cos\theta)^2$

$\Rightarrow \frac{d\sigma}{d(\cos\theta)} = \frac{G_F^2}{4\pi} s(1 + \cos\theta)^2$

Let $y = 1 + \cos\theta$

$dy = d(\cos\theta)$

$\frac{d\sigma}{dy} = \frac{G_F^2}{4\pi} s y^2$

so $\sigma = \int_{-1}^{1} \frac{G_F^2}{4\pi} s y^2 \, dy$

$= \frac{G_F^2}{6\pi} s$.

The cross section grows without restriction as $s \to \infty$.