Physics 564 - Fall 2005, Assignment #4 (preliminary)

The first part provides an introduction to the use of Mathematica to perform the evaluation of traces over gamma matrices and the simplification of the resulting products of 4-vectors. One of several packages developed for Mathematica is `Tracer`, which can be downloaded from the url

`http://physuna.phs.uc.edu/~johnson/HEP/mathematica/Tracer.m`

with documentation available at

`http://physuna.phs.uc.edu/~johnson/HEP/mathematica/tracer.ps`

Although developed to calculate the traces of higher order Feynman diagrams, its interface is convenient enough to illustrate its usefulness for evaluating the traces that arise in tree-level processes. Some examples and further explanation will be provided on the Physics 564 web page.

The documentation and the following example illustrate how to translate the notation we have used in class into the form used as input to Tracer. The invariant amplitude for the process $\mu^- \to e^- \overline{\nu}_e \nu_\mu$ was written

$$-i\mathcal{M} = -\frac{4G}{\sqrt{2}} J^\mu_{\text{muon}} J^{\text{electron}}_\mu \tag{1}$$

where

$$J^\mu_{\text{muon}} = \overline{u}_3 \frac{1}{2}\gamma^\mu (1-\gamma^5) u_1 \tag{2}$$

$$J^{\text{electron}}_\mu = \overline{u}_4 \frac{1}{2}\gamma_\mu (1-\gamma^5) v_2 \tag{3}$$

and the spin-averaged invariant amplitude squared was

$$|\overline{\mathcal{M}}| = \frac{G^2}{2} L^{\mu\nu}_{\text{muon}} L^{\text{electron}}_{\mu\nu} \tag{4}$$

where

$$L^{\mu\nu}_{\text{muon}} = \text{Tr}\left(\not{p}_3 \gamma^\mu \not{p}_1 \gamma^\nu (1-\gamma^5)\right) \tag{5}$$

$$L^{\text{electron}}_{\mu\nu} = 2\text{Tr}\left(\not{p}_4 \gamma_\mu \not{p}_2 \gamma_\nu (1-\gamma^5)\right) \tag{6}$$

The following Mathematica script will evaluate the traces and multiply the tensors to obtain a simplified expression for $|\overline{\mathcal{M}}|$:

```
<<Tracer.m                                       (* Load the Tracer package *)
VectorDimension[4]               (* We only need 4-dimensional gamma matrices  *)
AntiCommute[on]                            (* In 4-d, gamma5 will anti-commute *)
Spur[l]                                     (* Evaluate traces labeled by 'l' *)
lmuon = GammaTrace[l,p3,{mu},p1,{nu},(U-G5)]             (* Muon trace tensor *)
lelectron = 2 GammaTrace[l,p4,{mu},p2,{nu},(U-G5)]   (* Electron trace tensor *)
m = G^2/2 lmuon lelectron        (* Spin averaged squared invariant amplitude *)
Simplify[m]                                            (* Simplify the result *)
```

When this script is stored in the file `muondecay.m`, and used as input to Mathematica, the end result, after several lines of output, is:

```
$ math < muondecay.m

(skipping lots of output...)

              2
  Out[8]= 64 G  p1.p2 p3.p4
```

which is to be interpreted as

$$|\overline{\mathcal{M}}| \;=\; 64 G_F^2 p_1 \cdot p_2 p_3 \cdot p_4. \tag{7}$$

This is the result in the text and obtained in class. Note that the $(1-\gamma^5)$ is written as `(U-G5)` for input to Tracer, in which `U` is the 4×4 unit matrix. *The expression constructed using* `(1-G5)` *will give the wrong result.*

One can also use Mathematica to substitute the Mandlestam variables for the products of 4-vectors by adding the lines

```
%/.{p1.p2->s/2,p3.p4->s/2}                  (* Substitute in the previous result *)
%/.{s->mass^2}                                          (* More substitutions *)
```

to obtain

```
           2  2
Out[9]= 16 G  s
             2     4
Out[10]= 16 G  mass
```

We studied deep inelastic scattering experiments in which the internal structure of the proton was probed using high energy beams of electrons or neutrinos. The distribution of the spins of the quarks in the proton can be studied using a polarized electron beam and a polarized proton target.

1a. Calculate the differential cross section in the center-of-mass frame for a right-handed electron beam scattering of left- and right-handed quarks of flavor f, which have charge Q_f. Show that

$$\frac{d\hat{\sigma}_{++}}{d\Omega} = \frac{\alpha^2 Q_f^2 \hat{s}}{q^4} \tag{8}$$

$$\frac{d\hat{\sigma}_{+-}}{d\Omega} = \frac{\alpha^2 Q_f^2 \hat{s}}{q^4} \cos^4 \theta^*/2 \tag{9}$$

where $++$ indicates a right-handed electron colliding with a right-handed quark and $+-$ indicates a left-handed electron colliding with a left-handed quark. It is safe to neglect the mass of the electron.

1b. Use these expressions to show that the *spin-averaged* cross section can be written

$$\frac{d\hat{\sigma}}{d\Omega} = \frac{\alpha^2 Q_f^2 \hat{s}}{2q^4} \left(1 + \cos^4 \theta^*/2\right) \tag{10}$$

which is what we obtained previously in class.

1c. Express the cross sections in terms of the variable

$$y = \frac{E_e - E_e'}{E_e} \tag{11}$$

defined in the lab frame, as was done for other deep inelastic scattering processes.

1d. Use these cross sections for eq scattering to write expression for polarized ep and en scattering in terms of the parton density functions:

$u_p^+(x)$	u quark density in proton, spin aligned with proton spin	(12)
$u_p^-(x)$	u quark density in proton, spin aligned opposite proton spin	(13)
$d_p^-(x)$	d quark density in proton, spin aligned opposite proton spin	(14)
$d_p^+(x)$	d quark density in proton, spin aligned with proton spin	(15)

with similar ones for the neutron.

1e. Express the polarization asymmetry

$$\frac{\frac{d\sigma_{++}}{dxdy} - \frac{d\sigma_{+-}}{dxdy}}{\frac{d\sigma_{++}}{dxdy} + \frac{d\sigma_{+-}}{dxdy}} \tag{16}$$

in terms of differences in parton density functions, $\Delta u_p(x) = u_p^+(x) - u_p^-(x)$, *etc.*

1f. Assuming that the sea quarks are unpolarized, what would one naively expect to obtain for the polarization asymmetry?

1g. Experimentally, this result does not appear to be satisfied. What might this imply about the polarization of $s\bar{s}$ pairs or gluons in the proton? Early experimental results on this result were referred to as the “proton spin crisis” of 1989.

2. By considering the relationship between strong and weak eigenstates of the d and s quarks, show that the decay $K_L^0 \to \mu^+\mu^-$, mediated by a virtual Z^0 is forbidden.

3. The D^0 meson decays into $K^-\pi^+$, K^-K^+, $\pi^-\pi^+$ and π^-K^+ final states. Classify these as being Cabibbo allowed, Cabibbo suppressed or doubly-Cabibbo suppressed transitions and estimate their relative branching ratios.

4. Use the measured branching fractions for $\tau^- \to \pi^-\nu_\tau$ and $\tau^- \to K^-\nu_\tau$ to estimate the Cabibbo angle, θ_C.