Physics 564 - Fall 2005, Assignment #3

1a. The Dirac equation, $(i\gamma^\mu\partial_\mu - m)\psi = 0$ can describe the motion of an electron in the presence of an electromagnetic field with the substitution $\partial_\mu \to \partial_\mu - ieA_\mu$ where e is the *magnitude* of the electron charge. That is, an electron will have charge $Q_{e^-} = -e$ and a positron will have charge $Q_{e^+} = +e$. If $\psi(x)$ is written $\psi(x) = u(p)e^{-ip\cdot x}$, where $u(p)$ is expressed in terms of the two components

$$u(p) = \begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix}, \tag{1}$$

where u_A and u_B are 1×2 column vectors, show that they satisfy the coupled system of equations:

$$\begin{aligned} (E - m + eA^0)u_A - \vec{\sigma}\cdot(\vec{p} + e\vec{A})u_B &= 0 \quad (2) \\ \vec{\sigma}\cdot(\vec{p} + e\vec{A})u_A - (E + m + eA^0)u_B &= 0 \quad (3) \end{aligned}$$

.

1b. For a non-relativistic electron, write $E = m + E_{\rm NR}$ where $E_{\rm NR}$ is the non-relativistic kinetic energy, and use the fact that $E_{\rm NR} \ll m$ and $eA^0 \ll m$ to derive an equation for u_A alone.

1c. Use the identities

$$\begin{aligned} (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) &= i(\vec{\sigma}\times\vec{a})\cdot\vec{b} + \vec{a}\cdot\vec{b} = i\vec{\sigma}\cdot(\vec{a}\times\vec{b}) + \vec{a}\cdot\vec{b} \quad (4) \\ \vec{p}\times\vec{A} &= -i\nabla\times\vec{A} - \vec{A}\times\vec{p} \quad (5) \\ \vec{B} &= \nabla\times\vec{A} \quad (6) \end{aligned}$$

to show that

$$\left(\frac{|\vec{p} + e\vec{A}|^2}{2m} + \frac{e\vec{\sigma}\cdot\vec{B}}{2m} - eA^0\right)u_A = E_{\rm NR}u_A \tag{7}$$

1d. If the mangetic field, $\vec{B}$, is in the $+z$ direction, explain which spin states $u_A = \chi^{(s)}$ correspond to, where

$$\begin{aligned} \chi^{(1)} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (8) \\ \chi^{(2)} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (9) \end{aligned}$$

1e. The coupled equations (2,3) describe a non-relativistic positron with the substitutions $E \to -m - E_{\rm NR}$, $\vec{p} \to -\vec{p}$. Repeat the above steps to show that a non-relativistic positron satisfies the equation:

$$\left(\frac{|\vec{p} - e\vec{A}|^2}{2m} - \frac{e\vec{\sigma}\cdot\vec{B}}{2m} + eA^0\right)u_B = E_{\rm NR}u_B \tag{10}$$

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1f. If the mangetic field, $\vec{B}$, is in the $+z$ direction, explain which spin states the solutions $u_B = \chi^{(s)}$ correspond to for a non-relativistic positron.

The previous exercise demonstrated that in the non-relativistic limit, the basis $\chi^{(s)}$ was useful for describing the spin states that were quantized along the z-axis. In general, however, this is not always the most useful direction for describing the spin states of electrons and positrons. If $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is a unit vector, the operator that rotates the 2-component spinor $\chi^{(s)}$ into one in which the spin is aligned along the $\hat{n}$ direction can be written

$$R(\theta, \phi) = \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 & -e^{-i\phi/2}\sin\theta/2 \\ e^{i\phi/2}\sin\theta/2 & e^{i\phi/2}\cos\theta/2 \end{pmatrix}. \tag{11}$$

2a. Show that the 2-component spinors $\xi^{(s)}(\hat{n}) = R(\theta, \phi)\chi^{(s)}$ are eigenvectors of the operator $\vec{\sigma} \cdot \hat{n} = R(\theta, \phi)\sigma_3 R^\dagger(\theta, \phi)$.

2b. When $\hat{n}$ is in the direction of the momentum, that is, $\hat{n} = \hat{p} = \vec{p}/|\vec{p}|$, show that the Dirac spinors

$$u^{(s)}(p) = \begin{pmatrix} \sqrt{E+m}\,\xi^{(s)}(\hat{p}) \\ \frac{\hat{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\xi^{(s)}(\hat{p}) \end{pmatrix} \qquad v^{(s)}(p) = \begin{pmatrix} \frac{\hat{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\xi^{(s)}(\hat{p}) \\ \sqrt{E+m}\,\xi^{(s)}(\hat{p}) \end{pmatrix} \tag{12}$$

are eigenvectors of the operator $\vec{\Sigma} \cdot \hat{p}$, where

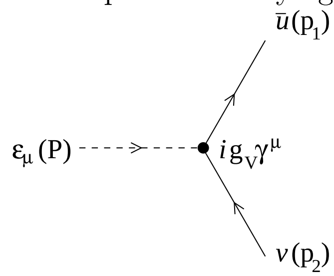
$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}. \tag{13}$$

Which eigenvalues correspond to electron and positron states with spins aligned parallel or anti-parallel to $\vec{p}$?

3a. Show that at high energies, $\gamma^5 u^{(s)}(p) \approx \vec{\Sigma} \cdot \hat{p} u^{(s)}(p)$.

3b. Particles with their spin aligned parallel to their momentum are called right-handed, while those with their spins aligned anti-parallel to their momentum are called left-handed. Thus, $u_R = u^{(1)}(p)$ and $u_L = u^{(2)}(p)$ describe right- and left-handed electrons, while $v_R = v^{(2)}(p)$ and $v_L = v^{(1)}(p)$ describe right- and left-handed positrons. Show that the projection operator $\frac{1}{2}(1+\gamma^5)$ projects out right-handed electron and left-handed positron states, while $\frac{1}{2}(1-\gamma^5)$ projects out left-handed electron and right-handed positron states.

The Feynman rules for a massive vector particle decaying to two fermions are indicated below:



Where g_V is a phenomenological coupling constant.

4a. Show that

$$\Gamma(V \to f\overline{f}) = \frac{g_V^2}{4\pi}\frac{|\vec{p}_f|}{M}(M^2 + 8/3m_f^2) \tag{14}$$

where M is the mass of the vector particle and m_f is the fermion mass. To average over the initial polarization of the meson, use the relation

$$\sum_{\text{spin}} \epsilon_\mu^*(P)\epsilon_\nu(P) = -g_{\mu\nu} + \frac{P_\mu P_\nu}{M^2} \tag{15}$$

.

4b. The $\psi(2S)$ meson can decay to e^+e^-, $\mu^+\mu^-$ and $\tau^+\tau^-$. Assuming that the coupling constant g_V does not depend on the type of fermion involved, calculate the ratios of partial widths, $\Gamma_{\mu\mu}/\Gamma_{ee}$ and $\Gamma_{\tau\tau}/\Gamma_{ee}$ and compare them with the experimental measurements listed in the Particle Data Book.