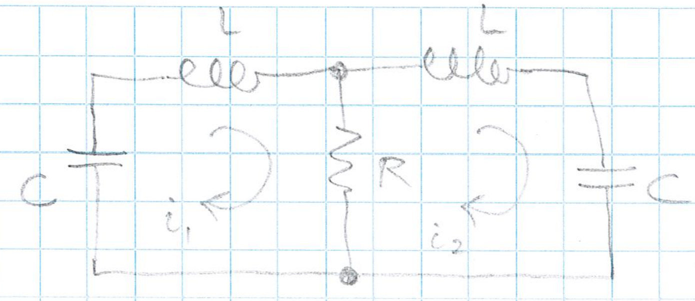
Assignment # 5 Phys 422 Spring '6 (1)

1.



(a) From Kirchoff's loop rule, the sum of the potential differences across the elements in each loop must be zero:

$$-L\frac{di_1}{dt} - R(i_1 - i_2) - \frac{1}{C}\int i_1\,dt = 0$$

$$-L\frac{di_2}{dt} - R(i_2 - i_1) - \frac{1}{C}\int i_2\,dt = 0$$

Differentiate each equation once more with respect to time:

$$-L\frac{d^2 i_1}{dt^2} - R\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) - \frac{1}{C}\,i_1 = 0$$

$$-L\frac{d^2 i_2}{dt^2} - R\left(\frac{di_2}{dt} - \frac{di_1}{dt}\right) - \frac{1}{C}\,i_2 = 0$$

If we write $\omega_0^2 = \frac{1}{LC}$ and $\gamma = \frac{R}{L}$

then these can be written:

$$\frac{d^2 i_1}{dt^2} + \gamma\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) + \omega_0^2\, i_1 = 0$$

$$\frac{d^2 i_2}{dt^2} + \gamma\left(\frac{di_2}{dt} - \frac{di_1}{dt}\right) + \omega_0^2\, i_2 = 0$$

(b) Assuming we can express the currents in the form

$$i_1(t) = Ae^{\alpha t} \quad \text{and} \quad i_2(t) = Be^{\alpha t}$$

we can write $\frac{di_1}{dt} = \alpha i_1$, $\frac{di_2}{dt} = \alpha i_2$

and $\frac{d^2 i_1}{dt^2} = \alpha^2 i_1$, $\frac{d^2 i_2}{dt^2} = \alpha^2 i_2$.

Thus, $$\alpha^2 i_1 + \alpha\gamma(i_1 - i_2) + \omega_0^2 i_1 = 0$$

$$x_2 i_2 + \alpha\gamma(i_2 - i_1) + \omega_0^2 i_2 = 0$$

(c) This system of equations can be written as a matrix equation:

$$\begin{pmatrix} \alpha^2 + \alpha\gamma + \omega_0^2 & -\alpha\gamma \\ -\alpha\gamma & \alpha^2 + \alpha\gamma + \omega_0^2 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = 0$$

For this to have non-trivial solutions, we must have

$$\det \begin{pmatrix} \alpha^2 + \alpha\gamma + \omega_0^2 & -\alpha\gamma \\ -\alpha\gamma & \alpha^2 + \alpha\gamma + \omega_0^2 \end{pmatrix} = 0$$

and so α must be a root of the polynomial

$$(\alpha^2 + \alpha\gamma + \omega_0^2)^2 - \alpha^2\gamma^2 = 0.$$

or $$\alpha^4 + 2\alpha^3\gamma + \cancel{\alpha^2\gamma^2} + 2\alpha\gamma\omega_0^2 + 2\alpha^2\omega_0^2 + \omega_0^4 \cancel{- \alpha^2\gamma^2} = 0$$

$$\alpha^4 + 2\alpha^3\gamma + 2\alpha^2\omega_0^2 + 2\alpha\gamma\omega_0^2 + \omega_0^4 = 0.$$

(d) If $i_1(t) = i_2(t)$ then there would be no net current flowing through the resistor and the set of differential equations would be just

$$\frac{d^2 i_1}{dt^2} + \omega_0^2 i_1 = 0$$

$$\frac{d^2 i_2}{dt^2} + \omega_0^2 i_2 = 0$$

which is satisfied when $\alpha = \pm i\omega_0 = \frac{\pm i}{\sqrt{LC}}$.

(e) We can divide out the factor $(\alpha^2 + \omega_0^2)$ from the characteristic polynomial using synthetic division:

$$\begin{array}{r|l}
 & \alpha^2 + 2\alpha\gamma + \omega_0^2 \\
\alpha^2 + \omega_0^2 & \alpha^4 + 2\alpha^3\gamma + 2\alpha^2\omega_0^2 + 2\alpha\gamma\omega_0^2 + \omega_0^4 \\
 & \underline{\alpha^4 + \alpha^2\omega_0^2} \\
 & 2\alpha^3\gamma + \alpha^2\omega_0^2 + 2\alpha\gamma\omega_0^2 + \omega_0^4 \\
 & \underline{2\alpha^3\gamma + 2\alpha\gamma\omega_0^2} \\
 & \alpha^2\omega_0^2 + \omega_0^4 \\
 & \underline{\alpha^2\omega_0^2 + \omega_0^4} \\
 & 0
\end{array}$$

Thus, we know that the other two roots are

$$\alpha = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$= -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2}.$$

(f) The frequencies of the normal modes of oscillation will be

$$\omega = \omega_0 = 1/\sqrt{LC}$$

$$\text{or} \quad \omega = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}.$$

2. $$y(x,t) = \sum_{n=1}^{\infty}\left(a_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t) + b_n \sin\left(\frac{n\pi x}{L}\right)\sin(\omega_n t)\right)$$

The initial displacement is

$$y(x,0) = f(x)$$

and the initial velocity is

$$\dot{y}(x,0) = g(x).$$

We need to make use of the important result:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}\delta_{mn}.$$

To solve for a_n, at time $t=0$ we have,

$$y(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x).$$

Thus, $$\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \sum_{m=1}^{\infty} a_m \int_0^L \sin\left(\frac{m\pi x}{2}\right)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \sum_{m=1}^{\infty} a_m\left(\frac{L}{2}\delta_{mn}\right)$$

$$= \frac{L}{2}a_n.$$

Hence, $$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

To solve for b_n, observe that at $t=0$ we have

$$\dot{y}(x,0) = \sum_{n=1}^{\infty} b_n \omega_n \sin\left(\frac{n\pi x}{L}\right) = g(x) .$$

Thus, $\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$$= \sum_{m=1}^{\infty} b_m \omega_m \int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \sum_{m=1}^{\infty} b_m \omega_m \left(\frac{L}{2} \delta_{mn}\right)$$

$$= \frac{L}{2} b_n \omega_n .$$

Thus, $b_n = \dfrac{2}{L\omega_n} \displaystyle\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$.

3. At $t = 0$ the pulse is described by

$$f(x) = \begin{cases} 0 & \text{for } x < 2/5 \\ 1 - 100\left(x - \frac{1}{2}\right)^2 & \text{for } 2/5 < x < 3/5 \\ 0 & \text{for } x > 3/5 \end{cases}.$$

If the pulse moves in the $+x$ direction with speed v then the shape of the string at $t = 0$ can be described by

$$y(x,t) = \begin{cases} 0 & \text{for } x < 2/5 \\ 1 - 100\left(x - vt - \frac{1}{2}\right)^2 & \text{for } 2/5 < x < 3/5 \\ 0 & \text{for } x > 3/5 \end{cases}.$$

The initial transverse velocity is just

$$\dot{y}(x,0) = \frac{\partial}{\partial t}\left(y(x,t)\right)\Big|_{t=0}$$

$$= -100 \cdot 2 \cdot \left(x - vt - \tfrac{1}{2}\right) \frac{\partial}{\partial t}\left(x - vt - \tfrac{1}{2}\right)$$

$$= 200\, v \left(x - vt - \tfrac{1}{2}\right)$$

$$= 200\, v \left(x - \tfrac{1}{2}\right) \quad \text{at } t = 0.$$

Hence

$$\dot{y}(x,0) = \begin{cases} 0 & \text{when } x < 2/5 \\ 200\, v\left(x - \frac{1}{2}\right) & \text{when } 2/5 < x < 3/5 \\ 0 & \text{when } x > 3/5 \end{cases}.$$