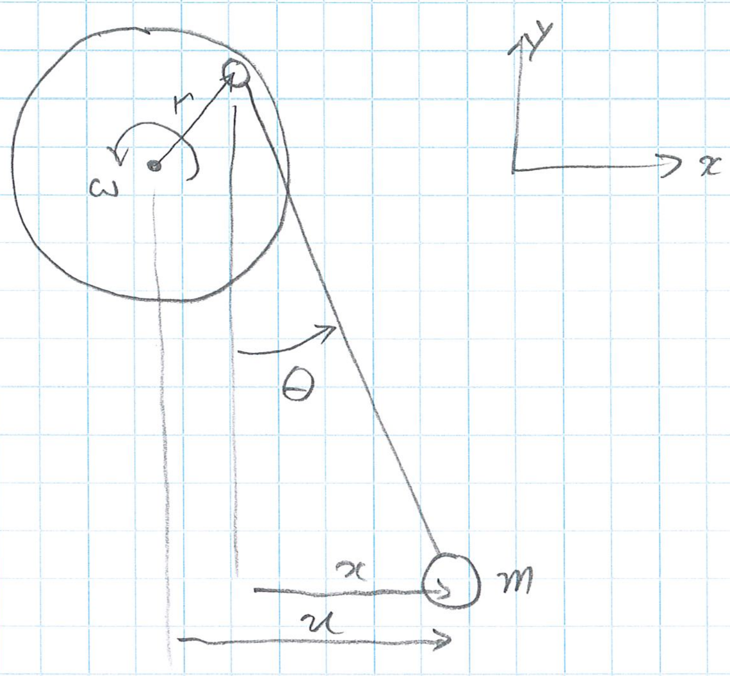
1.



(a) The wheel rotates with constant angular velocity ω so the coordinates of the pin, measured with respect to the axis of the wheel, are:

$$X(t) = r\cos\omega t$$
$$Y(t) = r\sin\omega t$$

(b) For small angles, θ, we can write

$$\theta \approx \sin\theta = \frac{x(t)}{L}.$$

However, $x(t)$ is measured in a non-inertial reference frame with respect to the moving pin. In the inertial reference frame we can write the following expression for the horizontal position of the mass:

$$u(t) = X(t) + x(t)$$

and $\quad v(t) = Y(t) + y(t) \quad$ for the vertical position.

If $y(t) = 0$ corresponds to the lowest point on the pendulum, then

$$y(t) = L(1 - \cos\theta) \approx 0 \quad \text{for small}$$

angles θ. Hence, $v(t) = Y(t)$ is just the same as the vertical position of the pin.

(c) The horizontal component of the tension is

$$F_x = -T\frac{x(t)}{L} \approx -\frac{mg}{L}x(t)$$

But this will cause an acceleration in the inertial reference frame:

$$m\frac{d^2u}{dt^2} = F_x$$

But $u(t) = X(t) + x(t)$ so

$$m\frac{d^2x}{dt^2} + \frac{mg}{L}x(t) = -m\frac{d^2X}{dt^2} = m\omega^2 r\cos\omega t$$

This is like the forced harmonic oscillator problem where $x(t)$ satisfies

$$\frac{d^2x}{dt^2} + \omega_0^2 x(t) = \frac{F_0}{m}\cos\omega t$$

where $F_0 = m\omega^2 r$ and $\omega_0^2 = \frac{g}{L}$

(d) With no forcing term the differential equation for $x(t)$ could be written

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

and solutions would be of the form

$$x(t) = A e^{-\gamma t/2} \cos(\omega' t + \varphi)$$

where $\omega' = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

If the amplitude decreases by $1/e$ in time τ then we must have

$$e^{-\gamma\tau/2} = e^{-1}$$

so $\gamma = 2/\tau$.

Then, when the wheel is turning $x(t)$ will satisfy

$$\frac{d^2x}{dt^2} + \frac{2}{\tau}\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

where $F_0 = mr\omega^2$.

For steady state oscillations the amplitude will be

$$A(\omega) = \frac{F_0/m}{\left((\omega_0^2-\omega^2)^2 + (\gamma\omega)^2\right)^{1/2}} = \frac{r\omega^2}{\left((\omega_0^2-\omega^2)^2 + \frac{4\omega^2}{\tau^2}\right)^{1/2}}$$

and $\tan\delta(\omega) = \dfrac{2\omega/\tau}{\omega_0^2 - \omega^2}$

(e) The Q-value is $Q = \frac{\omega_0}{\gamma} = \frac{2g}{L\tau}$.

The amplitude can be written in terms of Q as

$$A(\omega) = \frac{r\omega^2}{\left((\omega_0^2-\omega^2)^2 + (\omega\omega_0/Q)^2\right)^{1/2}}$$

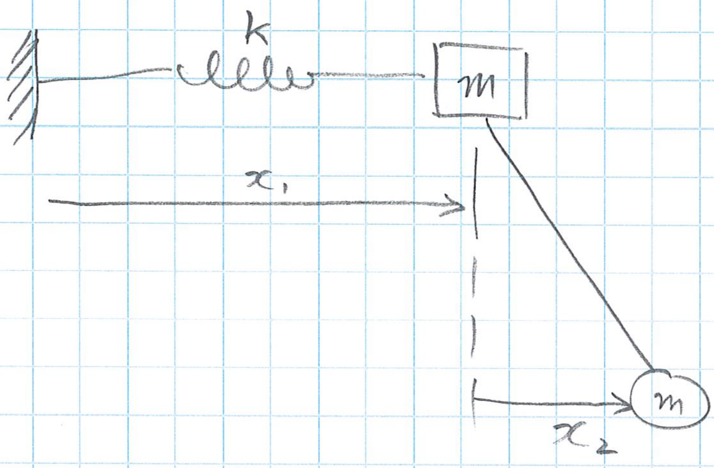
But when $\omega = \omega_0 = \sqrt{g/L}$ this is

$$A(\omega) = \frac{r\,\omega_0^2}{\left((\omega_0^2/Q)^2\right)^{1/2}}$$

$$= \frac{r\,\omega_0^2}{\omega_0^2/Q}$$

$$= rQ.$$

2.



(a)

The force acting on the top mass is

$$F_1 = -kx_1 + \frac{mg}{L} x_2$$

x_1 is already in an inertial reference frame, so

$$m\ddot{x}_1 = F_1 \Rightarrow \ddot{x}_1 + \frac{k}{m} x_1 - \frac{g}{L} x_2 = 0 .$$

The force acting on the bottom mass is

$$F_2 = -\frac{mg}{L} x_2$$

but x_2 is not in an inertial reference frame. Instead, $x_1 + x_2$ describes the position of the bottom mass in an inertial reference frame.

Thus, $$m(\ddot{x}_1 + \ddot{x}_2) = F_2$$

or $$\ddot{x}_1 + \ddot{x}_2 + \frac{g}{L} x_2 = 0 .$$

(b) When $k = mg/L$ the differential equations can be written

$$\ddot{x}_1 + \frac{g}{L}x_1 - \frac{g}{L}x_2 = 0$$

$$\ddot{x}_1 + \ddot{x}_2 + \frac{g}{L}x_2 = 0$$

Let $\omega_0^2 = g/L$. Then if we assume that

$$x_1 = A\cos\omega t$$
$$x_2 = B\cos\omega t$$

then
$$\ddot{x}_1 = -\omega^2 x_1$$
$$\ddot{x}_2 = -\omega^2 x_2$$

and therefore,

$$-\omega^2 x_1 + \omega_0^2 x_1 - \omega_0^2 x_2 = 0$$

$$-\omega^2 x_1 - \omega^2 x_2 + \omega_0^2 x_2 = 0$$

$$\begin{pmatrix} -\omega^2+\omega_0^2 & -\omega_0^2 \\ -\omega^2 & -\omega^2+\omega_0^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

If we write $\lambda = \omega^2$ then we must have

$$\begin{vmatrix} \omega_0^2-\lambda & -\omega_0^2 \\ -\lambda & \omega_0^2-\lambda \end{vmatrix} = 0$$

$$(\omega_0^2-\lambda)^2 - \lambda\omega_0^2 = \lambda^2 - 3\lambda\omega_0^2 + \omega_0^4 = 0$$

$$\Rightarrow \lambda = \omega^2 = \left(\frac{3}{2} \pm \frac{\sqrt{9-4}}{2}\right)\omega_0^2 = \frac{g}{2L}\left(3 \pm \sqrt{5}\right)$$