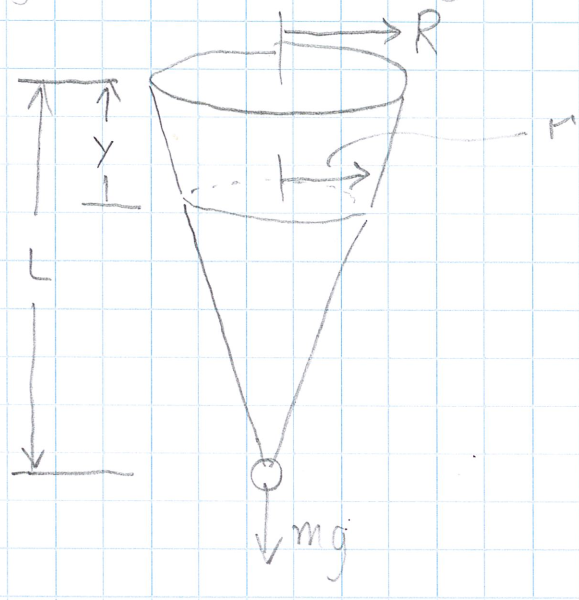
Assignment # 2

A conical float in water will be described using the following geometry:



(a) The volume of a cone with height h and base radius r is

$$V = \frac{\pi}{3} h r^2$$

In this case the length of the cone below the surface of the water is

$$h = L - y$$

and the radius at the waters surface is

$$r = \frac{R}{L}(L - y)$$

So the volume below the surface is

$$V(y) = \frac{\pi}{3} \frac{R^2}{L^2} (L - y)^3 = \frac{\pi}{3} R^2 L (1 - u)^3$$

where $u = y/L$.

(b) The total volume of the cone is

$$V_0 = \frac{\pi}{3} R^2 L$$

so the volume of displaced water can be written

$$V(y) = V_0 (1-u)^3$$
$$= V_0 (1 - 3u + 3u^2 - u^3)$$

where $u = y/L$.

The system is in equilibrium when the buoyant force and the force of gravity are equal.

Thus, $-mg + \rho g V(y) = 0$

or $\rho V(y) = \rho V_0 (1 - 3u + 3u^2 - u^3) = m$

But $\rho V_0 = M$ is the mass of water displaced by the entire cone. Hence,

$$1 - 3u + 3u^2 - u^3 = m/M.$$

The value of u that solves this equation will be a root of the cubic polynomial

$$1 - \frac{m}{M} - 3u + 3u^2 - u^3 = 0.$$

There are various ways to find roots of a cubic equation. One method is apparently due to Girolamo Cardano (1501 – 1576):

If $ax^3 + bx^2 + cx + d = 0$,

first make the substitution $x = y - \frac{b}{3a}$.

In our case,

$$u^3 - 3u^2 + 3u - \left(1 - \frac{m}{M}\right) = 0 .$$

Let $u = x - b/3a = x + 1$.

Then $(x+1)^3 - 3(x+1)^2 + 3(x+1) - \left(1 - \frac{m}{M}\right) = 0$

$$x^3 + \cancel{3x^2} + \cancel{3x} + \cancel{1} - \cancel{3x^2} - \cancel{6x} - \cancel{3} + \cancel{3x} + \cancel{3} - \cancel{1} + \frac{m}{M} = 0$$

$$x^3 = -\frac{m}{M} \quad \text{so} \quad x = -\sqrt[3]{\frac{m}{M}}$$

and $u_0 = 1 - \sqrt[3]{\frac{m}{M}}$.

Hence, when in equilibrium,

$$y_0 = u_0 L = L\left(1 - \sqrt[3]{\frac{m}{M}}\right) .$$

(You could also use Mathematica)

(c) If the float is displaced to $y = y_0 + \Delta y$, or equivalently, to $u = u_0 + \Delta u$ then the force will be

$$F = -mg + Mg\left(1 - 3(u_0 + \Delta u) + 3(u_0 + \Delta u)^2 - (u_0 + \Delta u)^3\right)$$

$$= -mg + Mg\left(1 - 3u_0 + 3u_0^2 - u_0^3\right)$$

$$+ Mg\left(-3\Delta u + 6u_0 \Delta u - 3u_0^2 \Delta u + \mathcal{O}(\Delta u^2)\right)$$

But the first term vanishes leaving

$$F(\Delta u) = -3Mg\left(1 - 2u_0 + u_0^2\right)\Delta u$$

$$= -3Mg\left(1 - u_0\right)^2 \Delta u$$

$$= -3Mg\left(1 - 1 + \sqrt[3]{\frac{m}{M}}\right)^2 \frac{\Delta y}{L}$$

$$= -\frac{3Mg}{L}\left(\frac{m}{M}\right)^{2/3} \Delta y$$

This is the same as Hooke's law

$$F = -kx \quad \text{with} \quad k = \frac{3Mg}{L}\left(\frac{m}{M}\right)^{2/3}$$

(d)

The frequency of small oscillations is

$$\omega^2 = \frac{k}{m} = \frac{3g}{L}\frac{M}{m}\left(\frac{m}{M}\right)^{2/3}$$

$$= \frac{3g}{L}\left(\frac{M}{m}\right)^{1/3}.$$

FORM C
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