Assignment # 4

1 (a) The Euler-Bernoulli equation in the absence of an external force is

$$YI \frac{\partial^4 w}{\partial x^4} + \mu \frac{\partial^2 w}{\partial t^2} = 0$$

If the boundary conditions are $w(0,t) = w(L,t) = 0$ then we propose that solutions are of the form

$$w(x,t) = A \sin(k_n x)\, e^{i\omega_n t} .$$

In order to satisfy the boundary condition $w(L,t) = 0$ we must have

$$k_n L = n\pi .$$

Therefore, $$k_n = \frac{n\pi}{L} .$$

(b) We need to substitute the proposed solution into the differential equation

$$\frac{\partial^4 w}{\partial x^4} = k_n^4\, w(x,t) \qquad \frac{\partial^2 w}{\partial t^2} = -\omega_n^2\, w(x,t)$$

Therefore, $$YI \frac{\partial^4 w}{\partial x^4} + \mu \frac{\partial^2 w}{\partial t^2} = \left(YI k_n^4 - \mu \omega_n^2 \right) w = 0$$

Hence, $$\omega_n^2 = \frac{YI}{\mu} k_n^4 = \frac{YI}{\mu} \left(\frac{n\pi}{L} \right)^4$$

$$\Rightarrow \omega_n = \pm \sqrt{\frac{YI}{\mu}} \left(\frac{n\pi}{L} \right)^2$$

(c) Using the constants:

$$Y = 69 \times 10^9 \ N \cdot m^{-2}$$
$$\rho = 2.7 \times 10^6 \ kg \cdot m^{-3}$$
$$L = 1 \ mm = 10^{-3} \ m$$
$$d = 25 \ \mu m = 25 \times 10^{-6} \ m$$

The radius of the wire is $r = 12.5 \times 10^{-6}$ m.
The area moment of inertia is

$$I = \frac{1}{2}\pi r^4 = \frac{1}{2}(12.5 \times 10^{-6} \ m) \times \pi = 3.835 \times 10^{-20} \ m^4$$

The mass per unit length is

$$\mu = \pi r^2 \rho = \pi (12.5 \times 10^{-6} \ m)^2 (2.7 \times 10^6 \ kg \, m^{-3})$$
$$= 1.325 \times 10^{-3} \ kg/m$$

Then $f_1 = \dfrac{\omega_1}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{YI}{\mu}}\left(\dfrac{\pi}{L}\right)^2$

$$= \frac{\pi}{2}\left(\frac{(69 \times 10^9 \ N \cdot m^{-2})(3.835 \times 10^{-20} \ m^4)}{1.325 \times 10^{-3} \ kg \cdot m^{-1}}\right)\left(\frac{\pi}{10^{-3} \ m}\right)^2$$

$$= 2220 \ s^{-1}$$

$$= 2.22 \ kHz.$$

2. This question asks for a solution to the initial value problem with boundary conditions $y(0,t) = y(L,t) = 0$ and initial conditions $y(x,0) = 0$, and

$$\dot{y}(x,0) = v\,x(L-x) \quad .$$

where v is just some constant with dimensions $m^{-1} \cdot s^{-1}$.

The general solution can be written

$$y(x,t) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\omega_n t\right)$$

$$\text{where } \omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$$

The choice of $\sin(\omega_n t)$ ensures that $y(x,0) = 0$.

The time derivative of the general solution is

$$\dot{y}(x,t) = \sum_n a_n \omega_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\omega_n t\right)$$

so at $t = 0$, $\dot{y}(x,0) = \sum_n a_n \omega_n \sin\left(\frac{n\pi x}{L}\right)$.

We can calculate $a_n \omega_n$ from the integral

$$a_n \omega_n = \frac{2}{L}\int_0^L \dot{y}(x,0) \sin\left(\frac{n\pi x}{L}\right) dx$$

As we showed in class (Lecture 16) the even terms must vanish.

The odd terms are symmetric about $x = L/2$ so we just need to calculate

$$a_n \omega_n = \frac{4v}{L}\int_0^{L/2} x(L-x)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$= 4v\int_0^{L/2} x \sin\left(\frac{n\pi x}{L}\right) dx$$

$$- \frac{4v}{L}\int_0^{L/2} x^2 \sin\left(\frac{n\pi x}{L}\right) dx$$

The first integral can be evaluated using

$$\int x \sin(ax)\, dx = -\frac{x}{a}\cos(ax) + \frac{1}{a^2}\sin(ax)$$

$$\int_0^{L/2} x \sin\left(\frac{n\pi x}{L}\right) dx = -\left(\frac{L}{2}\right)\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi}{2}\right) + \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)$$

But we only care about odd n, so

$$4v\int_0^{L/2} x \sin\left(\frac{n\pi x}{L}\right) dx = \pm 4v\left(\frac{L}{n\pi}\right)^2$$

The second integral can be evaluated using

$$\int x^2 \sin(ax)\, dx = \frac{2-a^2x^2}{a^3}\cos(ax) + \frac{2x}{a^2}\sin(ax)$$

So $$-\frac{4v}{L}\int_0^{L/2} x^2 \sin\left(\frac{n\pi x}{L}\right) dx = \mp\left(\frac{4v}{L}\right) L \cdot \left(\frac{L}{n\pi}\right)^2 + \frac{4v}{L}\cdot 2\left(\frac{L}{n\pi}\right)^3$$

The first terms from each integral cancel so we are left with

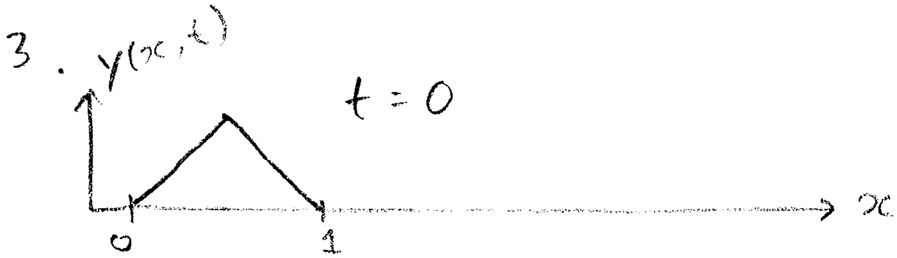
$$a_n \omega_n = \frac{8 v L^2}{(n\pi)^3} \quad \text{for } n = 1, 3, 5, 7, \ldots$$

Therefore, $a_n = 0$ when $n = 2, 4, 6, \ldots$

$$\text{and } a_n = \frac{8 v L^2}{(n\pi)^3} \cdot \frac{L}{(n\pi)} \sqrt{\frac{\mu}{T}} \quad \text{when } n = 1, 3, 5, \ldots$$

$$= \frac{8 v L^3}{(n\pi)^4} \sqrt{\frac{\mu}{T}}$$

3.



If the static pulse is described by the function $f(x)$ as shown, then the time dependent description of the pulse is

$$y(x,t) = f(x - vt) .$$

The transverse velocity is

$$\dot{y}(x,t) = \frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt} = -v f'(x - vt)$$

Since $f'(x) = \frac{0.4\,m}{0.5\,m} = 0.8$ for $0 < x < \frac{1}{2}$ m

and $f'(x) = -0.8$ for $\frac{1}{2} < x < 1$ m .

The derivative w. have the value of

$(24\,m/s)(0.8)$ for $0 < t < 0.5\,m / 24\,m/s$

and $-(24\,m/s)(0.8)$ for $\frac{0.5\,m}{24\,m/s} < t < \frac{1\,m}{24\,m/s}$.