Assignment #3

1. (a) The force on the left mass is

$$F_1 = -kx_1 - b(\dot{x}_1 - \dot{x}_2) = m\ddot{x}_1$$

The force on the right mass is

$$F_2 = -kx_2 - b(\dot{x}_2 - \dot{x}_1) = m\ddot{x}_2$$

The coupled set of differential equations is

$$m\ddot{x}_1 + kx_1 + b(\dot{x}_1 - \dot{x}_2) = 0$$
$$m\ddot{x}_2 + kx_2 + b(\dot{x}_2 - \dot{x}_1) = 0$$

which we can write

$$\ddot{x}_1 + \gamma\dot{x}_1 - \gamma\dot{x}_2 + \omega_0^2 x_1 = 0$$
$$\ddot{x}_2 + \gamma\dot{x}_2 - \gamma\dot{x}_1 + \omega_0^2 x_2 = 0$$

where $\omega_0^2 = k/m$ and $\gamma = b/m$.

(b) Let $x_i = A_i e^{\alpha t}$. Then $\dot{x}_i = \alpha x_i$ and $\ddot{x}_i = \alpha^2 x_i$.

Then
$$\alpha^2 x_1 + \alpha\gamma x_1 - \alpha\gamma x_2 + \omega_0^2 x_1 = 0$$
$$\alpha^2 x_2 + \alpha\gamma x_2 - \alpha\gamma x_1 + \omega_0^2 x_2 = 0$$

We can write this in matrix form as:

$$\begin{pmatrix} \alpha^2 + \alpha\gamma + \omega_0^2 & -\alpha\gamma \\ -\alpha\gamma & \alpha^2 + \alpha\gamma + \omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 .$$

(c) The determinant of the matrix is the characteristic polynomial:

$$(\alpha^2 + \alpha\gamma + \omega_0^2)^2 - \alpha^2\gamma^2 = 0$$

$$\alpha^4 + 2\gamma\alpha^3 + 2\omega_0^2\alpha^2 + 2\omega_0^2\gamma\alpha + \omega_0^4 = 0$$

(d) If the velocities of both masses were equal then the damper would exert no force on either mass. They would then behave as if they were only attached to their springs and would oscillate with frequency $\omega_0^2 = k/m$.

This would correspond to

$$\alpha^2 = -\omega_0^2$$

and consequently, $\alpha = \pm i\omega_0$ would be roots of the characteristic polynomial.

(e) Polynomial division

$$\begin{array}{r|lllll}
 & & & \alpha^2 & + 2\gamma\alpha & + \omega_0^2 \\ \hline
(\alpha^2+\omega_0^2) & \alpha^4 & + 2\gamma\alpha^3 & + 2\omega_0^2\alpha^2 & + 2\omega_0^2\gamma\alpha & + \omega_0^4 \\
 & \alpha^4 & & + \omega_0^2\alpha^2 & & \\ \cline{2-6}
 & & 2\gamma\alpha^3 & + \omega_0^2\alpha^2 & + 2\omega_0^2\gamma\alpha & + \omega_0^4 \\
 & & 2\gamma\alpha^3 & & + 2\omega_0^2\gamma\alpha & \\ \cline{3-6}
 & & & \omega_0^2\alpha^2 & & + \omega_0^4 \\
 & & & \omega_0^2\alpha^2 & & + \omega_0^4 \\ \cline{4-6}
 & & & & & 0
\end{array}$$

Therefore, the characteristic polynomial factors:

$$\alpha^4 + 2\gamma\alpha^3 + 2\omega_0^2\alpha^2 + 2\omega_0^2\gamma\alpha + \omega_0^4$$

$$= (\alpha^2 + \omega_0^2)(\alpha^2 + 2\gamma\alpha + \omega_0^2) = 0.$$

The other roots are obtained using the quadratic formula:

$$\alpha = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2}$$

$$= -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2}$$

(f) When $\alpha_1 = i\omega_0$ the eigenvectors satisfy

$$\begin{pmatrix} -\omega_0^2 + i\omega_0\gamma + \omega_0^2 & -i\omega_0\gamma \\ -i\omega_0\gamma & -\omega_0^2 + i\omega_0\gamma + \omega_0^2 \end{pmatrix}\begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} = 0$$

$$\begin{pmatrix} i\omega_0\gamma & -i\omega_0\gamma \\ -i\omega_0\gamma & i\omega_0\gamma \end{pmatrix}\begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} = 0$$

$$i\omega_0\gamma \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} = 0$$

$$u_1^{(1)} = u_2^{(1)}$$

If the eigenvector was normalized then we could write

$$\vec{u}^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The other eigenvalue $\alpha_2 = -i\omega_0$ corresponds to the same eigenvector:

$$\vec{u}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

These correspond to motion that is of the form

$$\vec{x}(t) = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_0 t} + B\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_0 t}$$

where A and B are constants of integration.

This could also be written

$$\vec{x}(t) = A\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_0 t + \theta)$$

When $\alpha_3 = -\gamma + i\sqrt{\omega_0^2 - \gamma^2}$

$$\begin{pmatrix} \gamma^2 - i\gamma\sqrt{\omega_0^2-\gamma^2} & \gamma^2 - i\gamma\sqrt{\omega_0^2-\gamma^2} \\ \gamma^2 - i\gamma\sqrt{\omega_0^2-\gamma^2} & \gamma^2 - i\gamma\sqrt{\omega_0^2-\gamma^2} \end{pmatrix}\begin{pmatrix} u_1^{(3)} \\ u_2^{(3)} \end{pmatrix} = 0$$

$$\left(\gamma^2 - i\gamma\sqrt{\omega_0^2-\gamma^2}\right)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u_1^{(3)} \\ u_2^{(3)} \end{pmatrix} = 0$$

$$u_1^{(3)} = -u_2^{(3)}$$

So we can write $\vec{u}^{(3)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Finally, when $\alpha_4 = -\gamma - i\sqrt{\omega_0^2 - \gamma^2}$

$$\begin{pmatrix} \gamma^2 + i\gamma\sqrt{\omega_0^2-\gamma^2} & \gamma^2 + i\gamma\sqrt{\omega_0^2-\gamma^2} \\ \gamma^2 + i\gamma\sqrt{\omega_0^2-\gamma^2} & \gamma^2 + i\gamma\sqrt{\omega_0^2-\gamma^2} \end{pmatrix}\begin{pmatrix} u_1^{(4)} \\ u_2^{(4)} \end{pmatrix} = 0$$

So again, $\vec{u}^{(4)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

This corresponds to motion of the form

$$\vec{x}(t) = A e^{-\gamma t} e^{it\sqrt{\omega_0^2-\gamma^2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

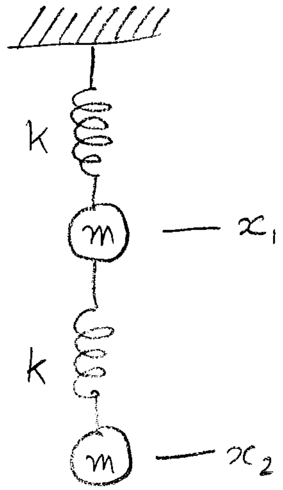
$$+ B e^{-\gamma t} e^{-it\sqrt{\omega_0^2-\gamma^2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

which can also be written in the form

$$\vec{x}(t) = A\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\gamma t}\cos(\omega t - \theta)$$

where $\omega = \sqrt{\omega_0^2 - \gamma^2}$.

2.



The force on the top mass is

$$F_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2 = m\ddot{x}_1$$

The force on the bottom mass is

$$F_2 = -k(x_2 - x_1) = -kx_2 + kx_1 = m\ddot{x}_2$$

These can be written

$$\ddot{x}_1 + 2\omega_0^2 x_1 - \omega_0^2 x_2 = 0$$
$$\ddot{x}_2 - \omega_0^2 x_1 + \omega_0^2 x_2 = 0$$

When $x_i(t)$ is of the form $x_i(t) = A\cos(\omega t + \varphi)$ then, $\ddot{x}_i = -\omega^2 x_i$ and the system of equations can be written

$$\begin{pmatrix} -\omega^2 + 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

The characteristic polynomial is

$$(-\omega^2 + 2\omega_0^2)(-\omega^2 + \omega_0^2) - \omega_0^4 = 0 .$$

or, $\lambda^2 - 3\omega_0^2 + \omega_0^4 = 0$ where $\lambda = \omega^2$.

The roots of the characteristic polynomial are

$$\lambda = \frac{3\omega_0^2 \pm \omega_0^2\sqrt{5}}{2} = \omega_0^2\left(\frac{3 \pm \sqrt{5}}{2}\right)$$

The eigenvector corresponding to the first root is determined from

$$\begin{pmatrix} 2\omega_0^2 - \omega_0^2\left(\frac{3+\sqrt{5}}{2}\right) & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega_0^2\left(\frac{3+\sqrt{5}}{2}\right) \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$\omega_0^2\begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ -1 & \frac{-1-\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

Thus, $\left(\frac{1-\sqrt{5}}{2}\right)u_1 - u_2 = 0$

$$\Rightarrow u_2 = \left(\frac{1-\sqrt{5}}{2}\right)u_1$$

The eigen vector corresponding to the second eigen value is

$$\begin{pmatrix} 2\omega_0^2 - \omega_0^2\left(\frac{3-\sqrt{5}}{2}\right) & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega_0^2\left(\frac{3-\sqrt{5}}{2}\right) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\omega_0^2\begin{pmatrix} \frac{1+\sqrt{5}}{2} & -1 \\ -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow v_2 = \left(\frac{1+\sqrt{5}}{2}\right)v_1$$

The general solution can be written

$$\vec{x}(t) = A\vec{u}\cos(\omega_1 t + \theta_1) + B\vec{v}\cos(\omega_2 t + \theta_2)$$

where $\omega_1 = \omega_0\sqrt{\frac{3+\sqrt{5}}{2}}$, $\omega_2 = \omega_0\sqrt{\frac{3-\sqrt{5}}{2}}$

and $\vec{u}$ and $\vec{v}$ are the eigenvectors.

When the bottom mass is pulled down by a force F, the initial displacements are

$$x_1(0) = F/k$$
$$x_2(0) = 2F/k \; .$$

and when the masses are released from rest, $\dot{x}_1(0) = \dot{x}_2(0) = 0$.

Thus, $A\begin{pmatrix} 1 \\ \frac{1}{2}(1-\sqrt{5}) \end{pmatrix}\cos\theta_1 + B\begin{pmatrix} 1 \\ \frac{1}{2}(1+\sqrt{5}) \end{pmatrix}\cos\theta_2 = \frac{F}{k}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$-A\omega_1\begin{pmatrix} 1 \\ \frac{1}{2}(1-\sqrt{5}) \end{pmatrix}\sin\theta_1 + B\omega_2\begin{pmatrix} 1 \\ \frac{1}{2}(1+\sqrt{5}) \end{pmatrix}\sin\theta_2 = 0$$

We expect that $\theta_1 = \theta_2 = 0$. Then

$$A + B = \frac{F}{k} \Rightarrow B = \frac{F}{k} - A$$

and

$$\tfrac{1}{2}A(1-\sqrt{5}) + \tfrac{1}{2}B(1+\sqrt{5}) = \frac{2F}{k}$$

$$\tfrac{1}{2}A(1-\sqrt{5}) + \tfrac{1}{2}(F/k - A)(1+\sqrt{5}) = \frac{2F}{k}$$

$$-A\sqrt{5} = \frac{2F}{k} - \frac{F}{2k}(1+\sqrt{5}) = \frac{F}{k}\left(\frac{3-\sqrt{5}}{2}\right)$$

$$A = \frac{F}{k}\left(\frac{5-3\sqrt{5}}{10}\right), \quad B = \frac{F}{k}\left(\frac{5+3\sqrt{5}}{10}\right)$$