Assignment #2

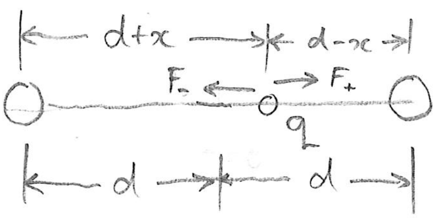
1 (a) The force exerted on a charge q in the presence of another charge Q is given by Coulomb's law:

$$F = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{r^2}$$

and is directed along the vector from Q to q.

When q is located at position $x(t)$ and two charges Q are located at $x = \pm d$, the force on q is

$$F = \frac{-1}{4\pi\varepsilon_0} \frac{qQ}{(d-x)^2} + \frac{1}{4\pi\varepsilon_0} \frac{qQ}{(d+x)^2}$$



If we write $F = -kx + O(x^2)$ then $k = -\left.\frac{dF}{dx}\right|_{x=0}$

$$\frac{dF}{dx} = \frac{-qQ}{4\pi\varepsilon_0} \cdot \frac{2}{(d-x)^3} - \frac{qQ}{4\pi\varepsilon_0} \cdot \frac{2}{(d+x)^3}$$

So $$-\left.\frac{dF}{dx}\right|_{x=0} = \frac{qQ}{\pi\varepsilon_0 d^3}$$

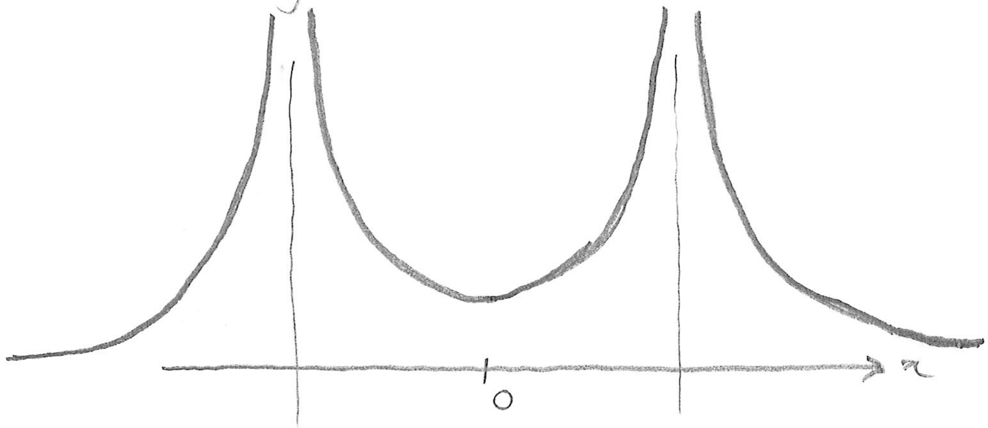
The equation of motion is approximately

$$m\frac{d^2x}{dt^2} = -kx = -\frac{qQ}{\pi\varepsilon_0 d^3}\, x$$

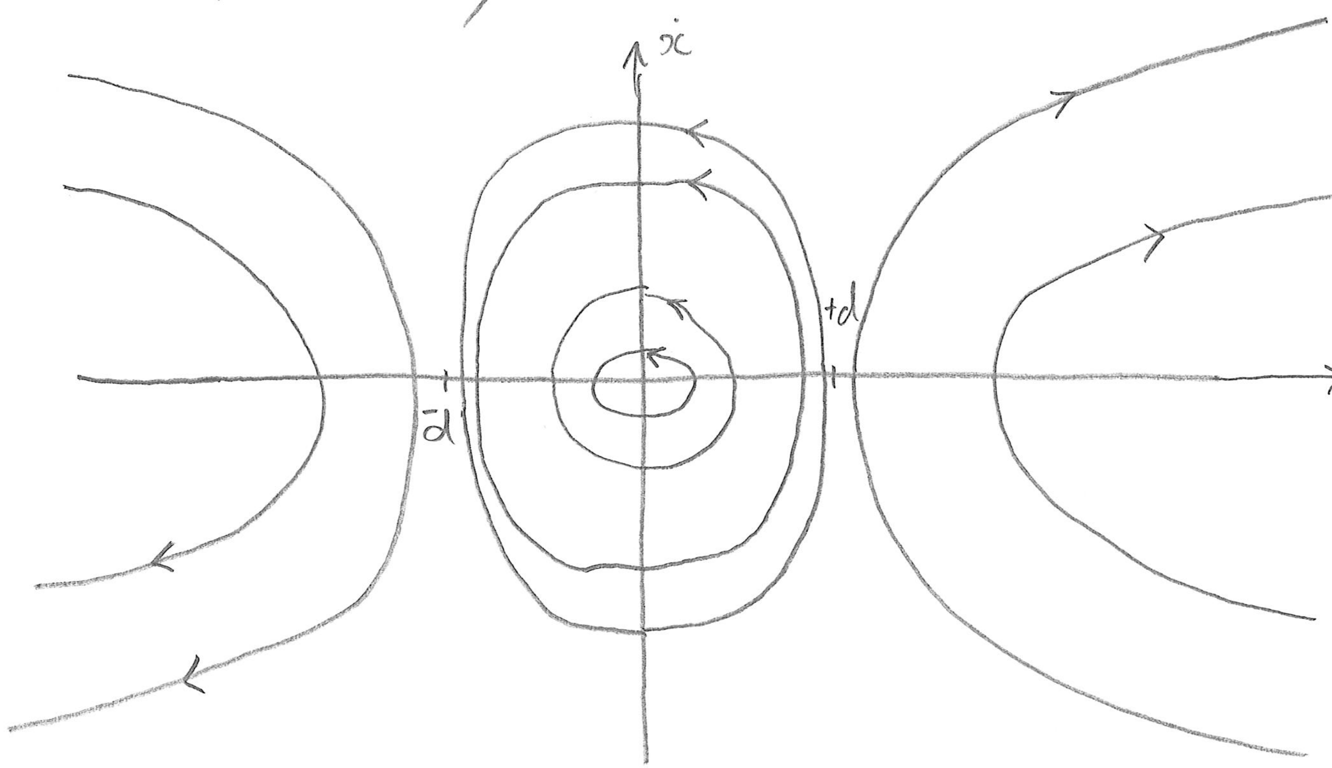
So $\ddot{x} + \omega_0^2 x = 0$ where $\omega_0 = \sqrt{\frac{qQ}{\pi\varepsilon_0 d^3 m}}$.

is the angular frequency of small oscillations.

(b) The potential energy diverges when $x \to \pm d$. Therefore, the graph of $V(x)$ looks something like this:



(c) The amplitude of oscillations will be bounded by $x = \pm d$.



2. $$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + (\omega\omega_0)^2/Q^2}}$$

When $A(\omega)$ is maximal, $\frac{dA}{d\omega} = 0$.

$$\frac{dA}{d\omega} = \frac{F_0}{m}\cdot\frac{-1/2}{\left((\omega_0^2-\omega^2)^2 + (\omega\omega_0)^2/Q^2\right)^{3/2}}\cdot\left(-4\omega(\omega_0^2-\omega^2) + 2\frac{\omega_0^2\omega}{Q^2}\right)$$

This vanishes when

$$\frac{\omega_0^2}{Q^2} = 2(\omega_0^2-\omega^2)$$

Hence, $$\omega^2 = \omega_0^2 - \frac{\omega_0^2}{2Q^2} = \omega_0^2\left(1-\frac{1}{2Q^2}\right)$$

So $$\omega = \omega_0\sqrt{1-\frac{1}{2Q^2}}$$

At this frequency, the amplitude is:

$$A(\omega) = \frac{F_0}{m}\cdot\frac{1}{\sqrt{\left(\omega_0^2-\omega_0^2+\frac{\omega_0^2}{2Q^2}\right)^2 + \frac{\omega_0^2}{Q^2}\left(\omega_0^2-\frac{\omega_0^2}{2Q^2}\right)}}$$

$$= \frac{F_0}{m}\frac{1}{\sqrt{\underbrace{\left(\frac{\omega_0^2}{2Q^2}\right)^2}_{1/4} \underbrace{-2\left(\frac{\omega_0^2}{2Q^2}\right)^2}_{-1/2} + \frac{\omega_0^4}{Q^2}}}$$

$$= \frac{F_0}{m}\cdot\frac{1}{\omega_0^2}\frac{Q}{\sqrt{1-1/4Q^2}}$$

$$= \frac{F_0}{k}\frac{Q}{\sqrt{1-1/4Q^2}}$$

3. For an undamped pendulum, the equation of motion is

$$\ddot{\theta} + \omega_0^2 \theta = 0$$

where $\omega_0 = \sqrt{g/l}$

When the motion is damped, the differential equation is

$$\ddot{\theta} + \gamma \dot{\theta} + \omega_0^2 \theta = 0$$

and solutions for underdamped oscillations are of the form

$$\theta(t) = A e^{-\gamma t/2} \cos(\omega t + \varphi)$$

where $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

When the amplitude is reduced by $1/e$ in time T we have

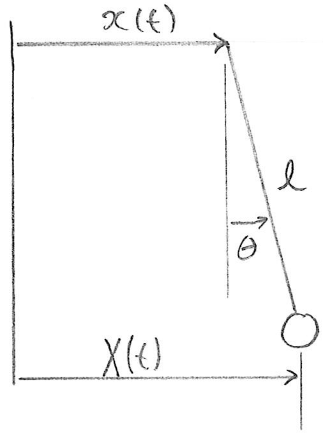
$$e^{-\gamma T/2} = e^{-1}$$

So $\frac{\gamma T}{2} = 1 \Rightarrow \gamma = \frac{2}{T}$.

The relation between γ and Q is

$$Q = \frac{\omega_0}{\gamma} = \frac{T}{2}\sqrt{g/l} .$$

4. Consider a pendulum that hangs as shown:



In this diagram, $X(t)$ is the position of the mass in an inertial reference frame. It is related to $x(t)$ and $\theta(t)$ by

$$X(t) = x(t) + l\,\theta(t) \qquad \text{when } \theta(t) \ll 1 .$$

The forces acting on the mass are gravity and the tension in the string. The force in the horizontal direction is

$$F = -mg\,\theta$$

So the equation of motion is

$$-mg\,\theta(t) = m\frac{d^2X}{dt^2} = m\frac{d^2x}{dt^2} + ml\frac{d^2\theta}{dt^2} .$$

which can be written

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta(t) = -\frac{1}{l}\frac{d^2x}{dt^2}$$

Including the effects of damping gives the differential equation

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega_0^2\theta = -\frac{1}{l}\frac{d^2x}{dt^2}$$

When $x(t) = d\cos\omega t$ this is

$$\ddot{\theta} + \gamma\dot{\theta} + \omega_0^2\theta = \frac{d\omega^2}{l}\cos\omega t$$

and the amplitude of forced oscillations is

$$A(\omega) = \frac{d\omega^2/l}{\sqrt{(\omega_0^2-\omega^2)^2-(\omega\gamma)^2}}.$$