Assignment # 1

1. (a) The roots of the polynomial

$$Ax^2 + Bx + C = 0$$

are just given by the quadratic equation:

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

(b) The roots will be real when $B^2 - 4AC > 0$.

2. Consider the complex valued function

$$z(t) = a e^{i\alpha} e^{i\omega t} + b e^{i\beta} e^{i\omega t}$$

which we would like to write in the form

$$z(t) = r e^{i(\omega t + \varphi)}$$

We can factor the $e^{i\omega t}$ so that

$$z(t) = \left(a e^{i\alpha} + b e^{i\beta} \right) e^{i\omega t}$$

in which case we just need to derive the conditions on r and φ such that

$$a e^{i\alpha} + b e^{i\beta} = r e^{i\varphi}.$$

First, $r^2 = |re^{i\varphi}|^2 = |ae^{i\alpha} + be^{i\beta}|^2$

$$= (ae^{i\alpha} + be^{i\beta})(ae^{-i\alpha} + be^{-i\beta})$$

$$= a^2 + ab\left(e^{i(\alpha-\beta)} + e^{i(\alpha-\beta)}\right) + b^2$$

Then, using the identity $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ this can be written:

$$r^2 = a^2 + b^2 + 2ab\cos(\alpha - \beta).$$

Since r is assumed to be positive we can write

$$r = \sqrt{a^2 + b^2 + 2ab\cos(a - \beta)}$$

The phase, φ, is determined as follows.
Since $e^{i\varphi} = \cos\varphi + i\sin\varphi = \frac{1}{r}\left(ae^{i\alpha} + be^{i\beta}\right)$

we can write
$$\tan\varphi = \frac{\sin\varphi}{\cos\varphi} = \frac{\operatorname{Im} e^{i\varphi}}{\operatorname{Re} e^{i\varphi}} = \frac{\operatorname{Im}(ae^{i\alpha} + be^{i\beta})}{\operatorname{Re}(ae^{i\alpha} + be^{i\beta})}.$$

Expand $ae^{i\alpha} + be^{i\beta}$ into real and imaginary parts:
$$ae^{i\alpha} + be^{i\beta} = a\left(\cos\alpha + i\sin\alpha\right) + b\left(\cos\beta + i\sin\beta\right)$$
$$= a\cos\alpha + b\cos\beta + i\left(a\sin\alpha + b\sin\beta\right)$$

Therefore,
$$\tan\varphi = \frac{a\sin\alpha + b\sin\beta}{a\cos\alpha + b\cos\beta}.$$

It is necessary to keep track of which quadrant φ lies in by examining the signs of the numerator and denominator.

3. (a) When two springs with spring constants k_1 and k_2 are connected in parallel, a force, F will stretch both springs by the same distance, Δx.

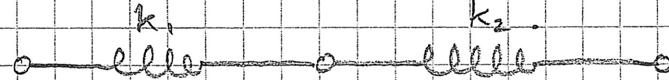
But, the force F is the sum of the forces exerted by the springs, so

$$F = F_1 + F_2 = -k_1 \Delta x - k_2 \Delta x$$
$$= -(k_1 + k_2)\,\Delta x$$
$$= -k_p\,\Delta x\,.$$

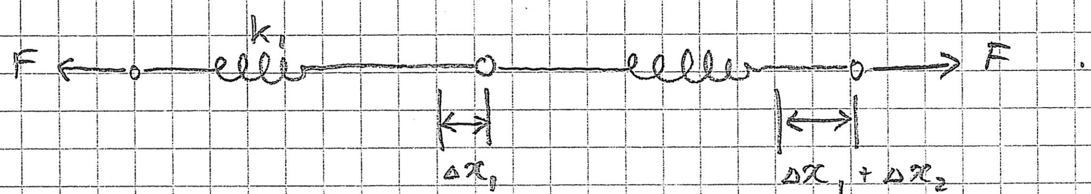
Thus, the effective spring constant for springs connected in parallel is

$$k_p = k_1 + k_2\,.$$

(b) Consider two springs that are connected in series as shown:



When a force F is applied to both springs, they will both stretch by amounts Δx_1 and Δx_2:



The force applied to each spring individually is still F, so

$$F = -k_1 \Delta x_1 = -k_2 \Delta x_2 = -k_s(\Delta x_1 + \Delta x_2)\,.$$

Thus, $\Delta x_1 = -\dfrac{F}{k_1}$, $\Delta x_2 = -\dfrac{F}{k_2}$

and $\Delta x_1 + \Delta x_2 = -F\left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right) = -\dfrac{F}{k_s}\,.$

Therefore, $k_s = \left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right)^{-1}\,.$

4. Consider an object of length L with cross sectional area given by the function $A(\ell)$ where $0 \le \ell \le L$.

If the object has uniform elastic modulus, Y then a thin slice of thickness $\Delta\ell$, located at a position ℓ will behave like a spring with spring constant k given by

$$k = \frac{YA(\ell)}{\Delta\ell}$$

so that it will stretch a distance x when subjected to a force F:

$$F = -\frac{YA(\ell)}{\Delta\ell}x = -kx.$$

(This result was derived in class or consider Eq 3-10 in the text.)

The entire object can be treated as many springs in series.

If the object is divided into N slices then, using the result from question 3 the effective spring constant will be

$$\frac{1}{k_{equiv}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_N}$$

In the limit as $\Delta\ell \to 0$ and $N \to \infty$ we can write this as an integral:

$$\frac{1}{k_{equiv}} = \int_0^L d\left(\frac{1}{k}\right)$$

$$\text{where } d\left(\frac{1}{k}\right) = \frac{d\ell}{YA(\ell)}$$

Therefore, $$\frac{1}{k_{equiv}} = \frac{1}{Y}\int_0^L \frac{d\ell}{A(\ell)}.$$

Whether or not this integral can be evaluated depends on the function $A(\ell)$... It could be done numerically of course.

5. One end of the spring is located at position

$$X(t) = \ell + vt$$

while the end with the mass attached to it is located at position $x(t)$. The force acting on the mass is then

$$F = -k(x(t) - X(t) + \ell)$$

The mass satisfies Newton's second law:

$$F = -k(x(t) - X(t) + \ell) = m\frac{d^2x}{dt^2}$$

Thus, $m\frac{d^2x}{dt^2} + kx(t) = k(X(t) - \ell) = kvt$.

Written in standard form this is

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{k}{m}vt \quad . \quad \text{where } \omega_0 = \sqrt{\frac{k}{m}}$$

The homogeneous equation is

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

and has solutions $x_1(t) = A\cos(\omega_0 t + \varphi)$.

A particular solution to the nonhomogeneous equation is

$$x_2(t) = \frac{1}{\omega_0^2} \cdot \frac{kvt}{m} = vt$$

Therefore, the complete solution to the differential equation is

$$x(t) = x_1(t) + x_2(t) = A\cos(\omega_0 t + \varphi) + vt$$

At time $t = 0$ the mass is located at $x = 0$.

Therefore we must have $\varphi = \pi/2$, and we can rewrite the solution in the form

$$x(t) = A\sin(\omega_0 t) + vt .$$

Also, at time $t=0$, the velocity of the mass is

$$\dot{x}(0) = 0$$

But, $\dot{x}(t) = A\omega_0 \cos \omega_0 t + v$

and at $t=0$ we must have

$$A\omega_0 + v = 0.$$

Therefore, $A = -v/\omega_0$ and the complete solution can be written

$$x(t) = -\frac{v}{\omega_0} \sin(\omega_0 t) + vt$$

Out of curiosity, we can check the direction of the acceleration.

$$\frac{dx}{dt} = -v\cos(\omega_0 t) + v$$

and $$\frac{d^2x}{dt^2} = v\omega_0 \sin(\omega_0 t)$$

which is positive for small $t \ll 1/\omega_0$.

Analyzing the problem this way is equivalent to recognising that $\bar{x}(t)$ could be used to represent the motion of the spring about its equilibrium position, which will be moving at constant velocity v.