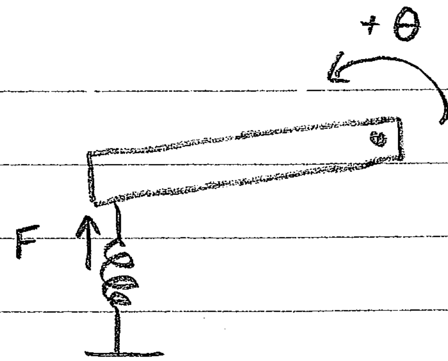
1.



Moment of inertia: $I = \frac{1}{3}ML^2$

Torque: $N = -FL = -kL^2\theta$

Equation of motion: $I\ddot{\theta} = N$

$$\frac{1}{3}ML^2\ddot{\theta} + kL^2\theta = 0$$

(a) $$\ddot{\theta} + \frac{3k}{m}\theta = 0$$

(b) $$\ddot{\theta} + \omega_0^2\theta = 0$$

where $\omega_0^2 = \frac{3k}{m}$

$\Rightarrow \omega_0 = \sqrt{3k/m}$.

2. $F_1 = -kx_1 + \frac{Mg}{\ell}x_2 = M\ddot{x}_1$

$$F_2 = -\frac{Mg x_2}{\ell} = M(\ddot{x}_1 + \ddot{x}_2)$$

$$= -kx_1 + \frac{Mg}{\ell}x_2 + M\ddot{x}$$

(a)

$$M\ddot{x}_1 + kx_1 - \frac{Mg}{\ell}x_2 = 0$$

$$M\ddot{x}_2 - kx_1 + \frac{2Mg}{\ell}x_2 = 0$$

(b) Let $\ddot{x}_1 = -\omega^2 x_1 = -\lambda x_1$

$$\ddot{x}_2 = -\omega x_2 = -\lambda x_2$$

Then $\begin{pmatrix} -\lambda + \omega_0^2 & -g/\ell \\ -\omega_0^2 & -\lambda + 2g/\ell \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$(\lambda - \omega_0^2)(\lambda - 2g/\ell) - \omega_0^2\, g/\ell = 0$$

$$\lambda^2 - \lambda(\omega_0^2 + 2g/l) + \omega_0^2 \, g/l = 0$$

$$\lambda = \frac{(\omega_0^2 + 2g/l) \pm \sqrt{(\omega_0^2 + 2g/l)^2 - 4\omega_0^2 \, g/l}}{2}$$

$$\omega^2 = \frac{\omega_0^2 + 2g/l \pm \sqrt{\omega_0^4 + 4g^2/l^2}}{2}$$

3. $y_k(t) = \sum_{n=1}^{N} a_n \sin\left(\frac{nk\pi}{N+1}\right) \cos \omega_n t$

$$y_k(0) = \sum_{n=1}^{N} a_n \sin\left(\frac{nk\pi}{N+1}\right) = A\,\delta_{mk}$$

$$\sum_{k=1}^{N} \sum_{n=1}^{N} a_n \sin\left(\frac{nk\pi}{N+1}\right) \sin\left(\frac{mk\pi}{N+1}\right)$$

$$= \sum_{k=1}^{N} y_k(0) \sin\left(\frac{mk\pi}{N+1}\right)$$

$$= \frac{N}{2} \sum_{n=1}^{N} a_n\, \delta_{nm} = \frac{N}{2} a_m$$

$$\text{So } a_n = \frac{2}{N} \sum_{k=1}^{N} y_k(0) \sin\left(\frac{nk\pi}{N+1}\right)$$

$$= \frac{2}{N} \sum_{k=1}^{N} A\,\delta_{mk} \sin\left(\frac{mk\pi}{N+1}\right)$$

$$= \frac{2A}{N} \sin\left(\frac{mn\pi}{N+1}\right)$$

4. The initial function is of the form

$$f(x) = \begin{cases} +b & \text{for } x < L/2 \\ -b & \text{for } x > L/2 \end{cases}.$$

The general form can be written

$$y(x,t) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2b}{L}\int_0^{L/2} \sin\left(\frac{n\pi x}{L}\right)dx$$

$$- \frac{2b}{L}\int_{L/2}^{L} \sin\left(\frac{n\pi x}{L}\right)dx$$

Let $y = L - x$. Then

$$a_n = \frac{2b}{L}\int_0^{L/2} \sin\left(\frac{n\pi x}{L}\right)dx + \frac{2b}{L}\int_{L/2}^{0} \sin\left(n\pi - \frac{n\pi y}{L}\right)d$$

$$= \frac{2b}{L}\int_0^{L/2} \sin\left(\frac{n\pi x}{L}\right)dx \mp \frac{2b}{L}\int_0^{L/2} \sin\left(\frac{n\pi y}{L}\right)d$$

$= 0$ when n is odd.

5. $$L\frac{di}{dt} + i(t)R + \frac{1}{C}\left(Q_0 + \int_0^t i(t)\,dt\right) =$$

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = 0$$

Let $i(t) = Ae^{\alpha t}$

then

$$\left(L\alpha^2 + R\alpha + \frac{1}{C}\right) i(t) = 0$$

$$\alpha = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$

Critical damping when $R^2 = 4L/C$

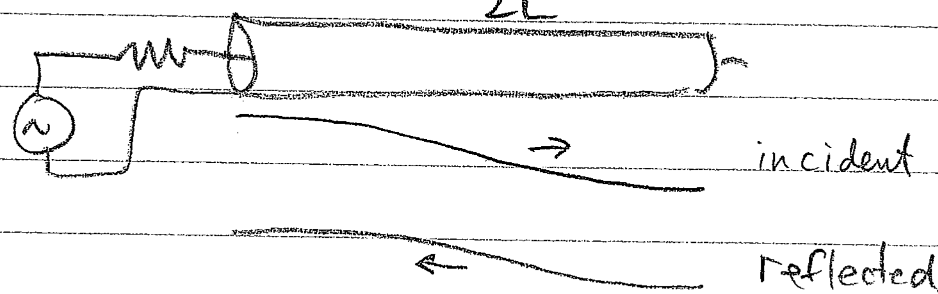
$$R = 2\sqrt{\frac{L}{C}}$$

6. (a) The reflection coefficient at the open end is $\rho = +1$.

In this case, a maximum is seen at A when

$$\lambda = 2L = \frac{v}{f}$$

$$\text{So } f = \frac{v}{2L}$$



(b) When $\rho = -1$, $\lambda = 4L$, $f = \frac{v}{4L}$