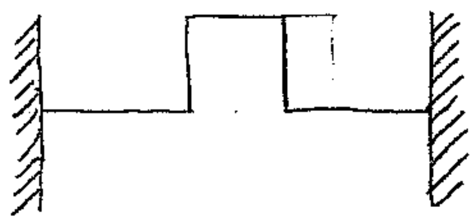
Assignment #4

1. The tension in the string is $T = 4\,N$ and the mass density is $\mu = 1\ kg/m$ so the speed of wave propagation is

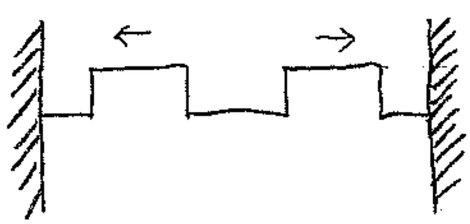
$$v = \sqrt{T/\mu} = 2\ m/s .$$

The width of the pulse is 2 m and when released from rest will evolve into two pulses, one moving to the left and one moving to the right.

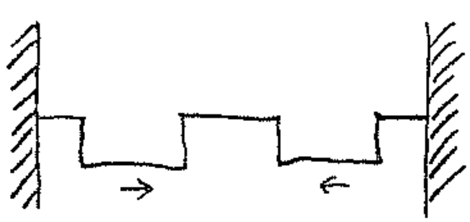
At $t=0$ the pulse looks like this:



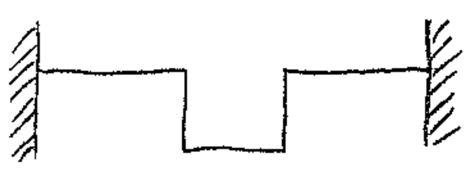
At time $t = 1s$, each pulse has moved 2 m:



At time $t = 3s$ the pulses have reflected from the ends, and are now inverted:



At $t = 4s$ the pulses overlap again:



2. The initial displacement of the string is zero everywhere while the initial velocity is

$$f(x) = u\,x(x-L) \quad .$$

The general solution to the wave equation with boundary conditions $y(0,t) = y(L,t) = 0$ can be expressed in terms of the normal modes of oscillation:

$$y(x,t) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t + \alpha_n) \quad .$$

At time $t=0$ the solution is

$$y(x,0) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right) \cos \alpha_n = 0 \quad .$$

Therefore, we must have $\cos \alpha_n = 0$ for all n which is satisfied by

$$\alpha_n = \pi/2 \quad .$$

Now we can write $\cos(\omega_n t + \pi/2) = -\sin \omega_n t$

The first derivative is now

$$\dot{y}(x,t) = -\sum_n a_n \omega_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

and at time $t=0$,

$$\dot{y}(x,0) = -\sum_n a_n \omega_n \sin\left(\frac{n\pi x}{L}\right) = u\,x(x-L) \quad .$$

which we can write as

$$\dot{y}(x,0) = \sum_n b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{where } b_n = -a_n \omega_n .$$

The coefficients b_n can be calculated from the expression

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\, dx$$

$$= \frac{2u}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) x(x-L)\, dx$$

$$= \frac{2u}{L}\int_0^L x^2 \sin\left(\frac{n\pi x}{L}\right) dx - 2u\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx$$

The first term can be integrated by parts:

Let $u = x^2 \qquad du = 2x\, dx$

$dv = \sin\left(\frac{n\pi x}{L}\right) \qquad v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

$$\int u\, dv = uv\Big|_0^L - \int v\, du$$

to obtain

$$b_n = -\frac{2ux^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{4u}{n\pi}\int_0^L x\cos\left(\frac{n\pi x}{L}\right) dx$$

$$- 2u\int_0^L x\sin\left(\frac{n\pi x}{L}\right) dx$$

The remaining integrals can be evaluated by integration by parts:

Let $u = x \qquad du = dx$

$dv = \cos\left(\frac{n\pi x}{L}\right) dx \qquad v = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$

$$\int u\, dv = uv\Big|_0^L - \int_0^L v\, du$$

$$b_n = -\frac{2ux^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{4uL}{n^2\pi^2}\, x \cancelto{0}{\sin\left(\frac{n\pi x}{L}\right)}\Big|_0^L$$

$$- \frac{4uL}{n^2\pi^2}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx + \frac{2uL}{n\pi}\, x\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

$$- \frac{2uL}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx$$

$$= -\frac{2ux^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{4uL^2}{n^3\pi^3}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{2uL}{n\pi}\, x\cos\left(\frac{n\pi x}{L}\right)\Big|$$

$$- \frac{2uL^2}{n^2\pi^2}\cancelto{0}{\sin\left(\frac{n\pi x}{L}\right)}\Big|_0^L$$

$$= -\frac{2uL^2}{n\pi}\cos(n\pi) + \frac{2uL^2}{n\pi}\cos(n\pi)$$

$$+ \frac{4uL^2}{n^3\pi^3}\left[\cos(n\pi) - 1\right]$$

$$= \frac{4uL^2}{n^3\pi^3}\left((-1)^n - 1\right)$$

But remember that $b_n = -a_n\omega_n$

where $\omega_n = \frac{n\pi v}{L}$.

Therefore, $a_n = -\frac{b_n}{\omega_n} = \frac{u}{v}\cdot\frac{4L^3}{n^4\pi^4}\left(1-(-1)^n\right)$

These coefficients are zero when n is even. When n is odd,

$$a_n = \frac{8uL^3}{vn^4\pi^4}.$$

3. The string on the left has mass density μ while the string on the right has mass density 4μ. The tension is the same in both strings so the velocities are

$$v_1 = \sqrt{T/\mu} \quad \text{and} \quad v_2 = \sqrt{T/4\mu} = \tfrac{1}{2} v_1 .$$

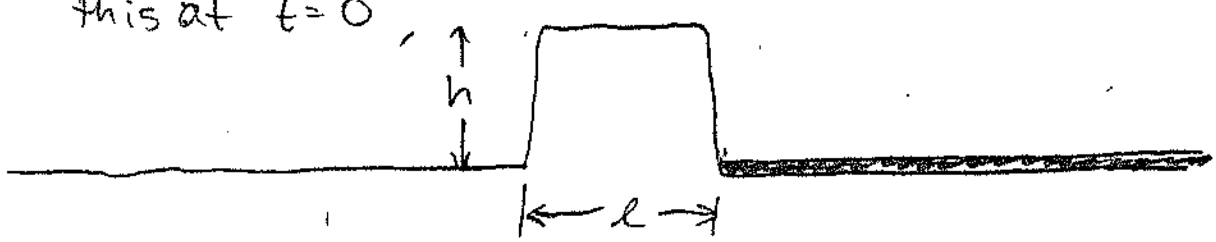
The reflection coefficient is

$$\rho = \frac{v_2 - v_1}{v_2 + v_1} = \frac{\tfrac{1}{2}v_1 - v_1}{\tfrac{1}{2}v_1 + v_1} = \frac{-1/2}{3/2} = -\frac{1}{3}$$

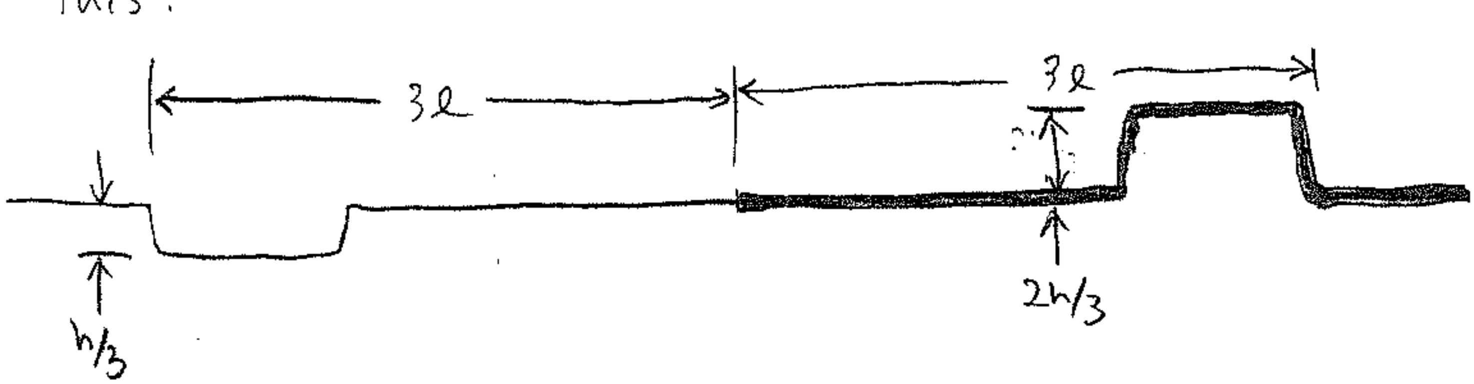
and the transmission coefficient is

$$\tau = \frac{2v_2}{v_2 + v_1} = \frac{2(\tfrac{1}{2}v_1)}{\tfrac{1}{2}v_1 + v_1} = \frac{1}{3/2} = \frac{2}{3}$$

At time $t = 3\ell/v$ the pulse will have completely reflected and the incident and reflected components would not interfere with each other. The initial pulse would look like this at $t=0$

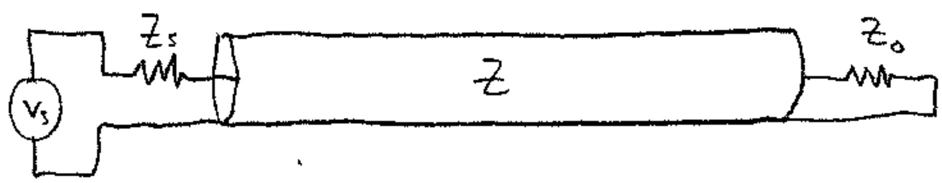


and at time $t = 3\ell/v$ the pulse would look like this:



4. The length of the cable is 30 m and signals propagate with a speed of 20 cm/ns. Therefore, it takes $T = \frac{3000\text{ cm}}{20\text{ cm/ns}} = 150\text{ ns}$ for signals to travel the length of the cable.

The circuit looks like this:



Where Z_s is the source impedance, Z is the characteristic impedance of the cable and Z_o is the impedance of the oscilloscope.

The reflection coefficient is

$$\rho = \frac{Z' - Z}{Z' + Z}$$

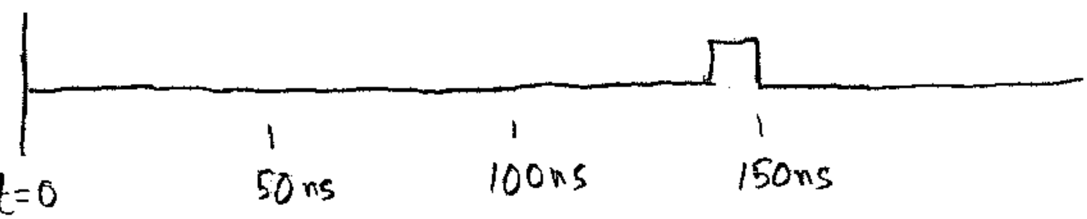
where Z' is either the source of the oscilloscope impedance.

We will assume that the voltage driven at the left end of the cable is V. This is not necessarily the same as the ideal source voltage V_s because of the voltage drop across Z_s. They are related by

$$V = \frac{Z V_s}{Z + Z_s}$$

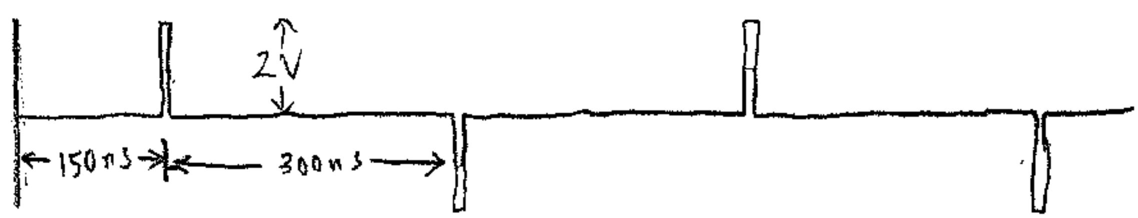
but this does not change the analysis of the reflections.

(a) When $Z_s = 0\,\Omega$ and $Z_o = 50\,\Omega$, the impedance of the oscilloscope matches the cable impedance, so the reflection coefficient is zero. The waveform would look like this:



(b) When $Z_o = 1\,M\Omega$ the reflection coefficient at the oscilloscope is 1 but when $Z_s = 0\,\Omega$, the reflection coefficient at the source is -1.

The waveform would look like this:



(c) This is the same as part (a). In part (a), the reflection coefficient at the source was -1 but since no pulse was reflected from the oscilloscope, none of the subsequent pulses would be observed.

(d) In this case the pulse height observed at the oscilloscope would be 2V because the reflected pulse is not inverted. But this pulse is not reflected at the source.

(e) The reflection coefficient at the oscilloscope is $\rho = 1$ and at the source it is

$$\rho = \frac{Z_0 - Z}{Z_0 + Z} = \frac{100\,\Omega - 50\,\Omega}{150\,\Omega} = \frac{1}{3}.$$

The waveform would look like this: