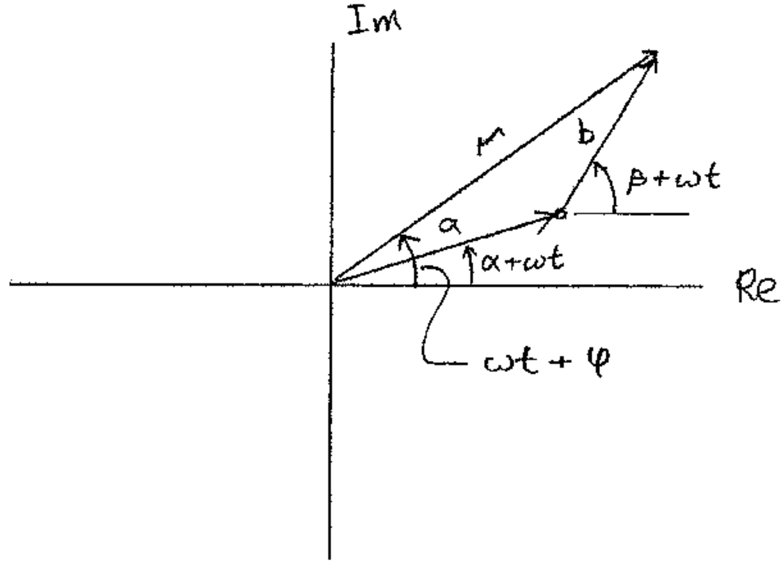
Assignment # 1.

1. Show that $z(t) = a e^{i\alpha} e^{i\omega t} + b e^{i\beta} e^{i\omega t}$ can be written as
$$z(t) = r e^{i(\omega t + \varphi)}$$
where $a, b, \alpha, \beta, r, \varphi$ and ω are real numbers.

This is easier to think about using phasors or by drawing vectors in the complex plane:



$$r = |z(t)| = \sqrt{z^* z}$$
$$= \left(\left(a e^{i(\omega t+\alpha)} + b e^{i(\omega t+\beta)}\right)^* \left(a e^{i(\omega t+\alpha)} + b e^{i(\omega t+\beta)}\right)\right)^{1/2}$$
$$= \left(\left(a e^{-i(\omega t+\alpha)} + b e^{-i(\omega t+\beta)}\right) \left(a e^{i(\omega t+\alpha)} + b e^{i(\omega t+\beta)}\right)\right)^{1/2}$$
$$= \left(a^2 + ab\left(e^{i(\alpha-\beta)} + e^{-i(\alpha-\beta)}\right) + b^2 \right)^{1/2}$$
$$= \left(a^2 + 2ab\cos(\alpha-\beta) + b^2 \right)^{1/2}$$

Recall that
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

The phase, φ, can be calculated by considering the case when $t=0$.

In this case,

$$\tan\varphi = \frac{\operatorname{Im} z}{\operatorname{Re} z}$$

When $t=0$,
$$\begin{aligned} z &= ae^{i\alpha} + be^{i\beta} \\ &= a(\cos\alpha + i\sin\alpha) + b(\cos\beta + i\sin\beta) \\ &= a\cos\alpha + b\cos\beta + i(a\sin\alpha + b\sin\beta) \end{aligned}$$

So $\operatorname{Im}(z) = a\sin\alpha + b\sin\beta$
and $\operatorname{Re}(z) = a\cos\alpha + b\cos\beta$.

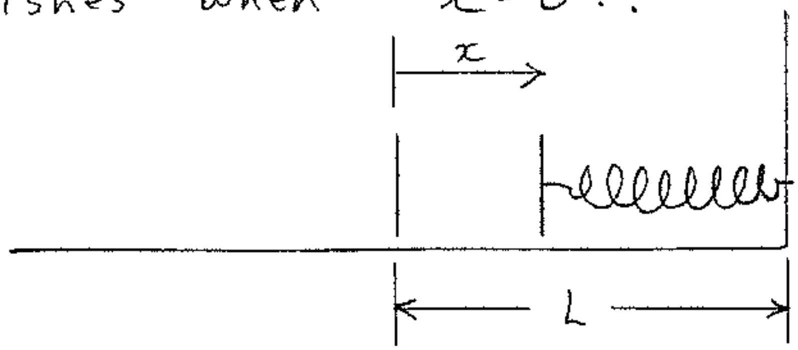
Thus, $$\tan\varphi = \frac{a\sin\alpha + b\sin\beta}{a\cos\alpha + b\cos\beta}$$

If you want to write a formal expression for φ itself, then

$$\varphi = \tan^{-1}\left(\frac{a\sin\alpha + b\sin\beta}{a\cos\alpha + b\cos\beta}\right)$$

2. Although there is no unique way to define the position, $x(t)$, some choices are more convenient than others. In this case, it is convenient to measure $x(t)$ with respect to the equilibrium position of the spring so that the force exerted by the spring vanishes when $x = 0$:.

(a)



x is positive when the spring is compressed and negative when the spring is stretched.

(b) The force exerted by the spring is

$$F(x) = -kx$$

With the mass stuck to the spring,

$$F(x) = m\ddot{x} = -kx$$

So $\quad m\ddot{x} + kx = 0$

or $\quad \ddot{x} + \omega^2 x = 0 \quad$ where $\omega = \sqrt{\frac{k}{m}}$.

Solutions are of the form

$$x(t) = A\sin\omega t + B\cos\omega t.$$

At time $t = 0$, $x = 0$ so $B = 0$.

The velocity is $\dot{x}(t) = A\omega \cos \omega t$ and at $t = 0$, $\dot{x}(t) = v_0$.

Thus, $A\omega = v_0$ so $A = v_0 / \omega$.

The solution is $x(t) = \frac{v_0}{\omega} \sin \omega t$ for $t > 0$.

3. When in equilibrium, the net force is zero. Thus

(a) $$F_s - mg = 0$$

where the sign convention is such that positive forces point up.

The amount the spring has been compressed is $L-z$ so $F_s = k(L-z)$, which points up when $z < L$.

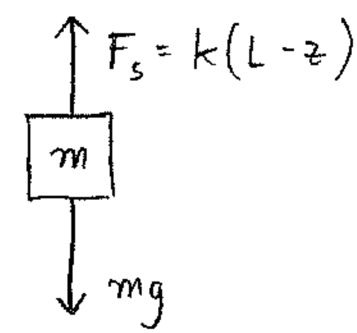
Hence, $$k(L-z_0) - mg = 0.$$

$$kL - mg = kz_0$$

So the equilibrium position is

$$z_0 = L - \frac{mg}{k}.$$

(b) The free-body diagram looks like this:



(c) The net force on the mass is

$$F = -mg + k(L-z) = -kz + kL - mg$$

(d) From Newton's second law, $F = m\ddot{z}$.

But then $$\ddot{z} + \omega^2 z = \frac{kL}{m} - g \neq 0.$$

so it is not of the form $\ddot{z} + \omega^2 z = 0$.

(e) Suppose that we measure distances with respect to the equilibrium position, z_0.

Then $z = z_0 + z'$.

The acceleration is $\ddot{z} = \ddot{z}'$

But $\omega^2 z = \frac{k}{m} z = \frac{k}{m}(z_0 + z')$

$$= \frac{k}{m}\left(L - \frac{mg}{k}\right) + \frac{k}{m} z'$$

So $\ddot{z} + \omega^2 z = \frac{kL}{m} - g$

$$\ddot{z}' + \frac{k}{m} z' + \left(\frac{kL}{m} - g\right) = \frac{kL}{m} - g$$

Therefore, $\ddot{z}' + \omega^2 z' = 0$.

(f) When the spring is uncompressed,

$$z = L = z_0 + z'$$

So $z' = L - z_0 = L - \left(L - \frac{mg}{k}\right)$

$$= \frac{mg}{k}.$$

Solutions to $\ddot{z}' + \omega^2 z' = 0$ are of the form

$$z'(t) = A \sin \omega t + B \cos \omega t.$$

When $t = 0$, $\dot{z}' = \frac{mg}{k}$ so $B = mg/k$.

When $t=0$, $\dot{z}' = A\omega$ but the mass was released from rest, so $\dot{z}'(t) = 0$ at $t=0$

Thus, $A=0$ so the solution is

$$z'(t) = \frac{mg}{k} \cos \omega t \qquad \text{where } \omega = \frac{k}{m}.$$

Expressed in terms of z,

$$z(t) = L - \frac{mg}{k} + z'(t)$$

$$= L - \frac{mg}{k}\left(1 - \cos \omega t \right)$$