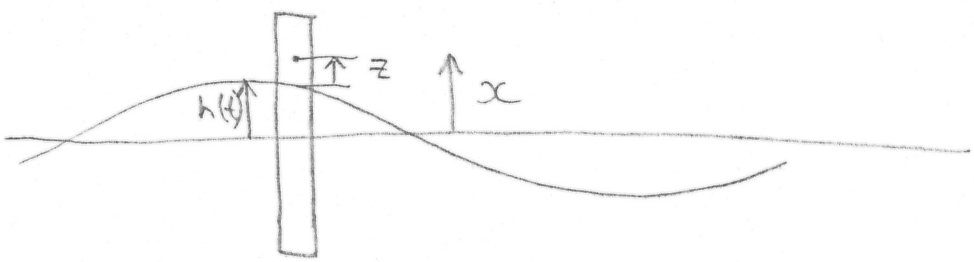
Assignment # 3

1. Consider the following reference frame in which the log is displaced by a distance z beyond its equilibrium position, measured from the surface of the water:



The buoyant force acting on the log is

$$F_b = -\rho g z a$$

but this is equal to M times the acceleration in an inertial reference frame.

If x is the position of the log in an inertial reference frame, then

$$F = M\ddot{x} = M(\ddot{h} + \ddot{z})$$

Hence, $M\ddot{z} + \rho g z a = -M\ddot{h}$.

If the waves have amplitude A and angular frequency ω, then we can write

$$h(t) = A \cos \omega t$$

and hence, $\ddot{h} = -A\omega^2 \cos \omega t$.

The equation of motion is then

$$M\ddot{z} + \rho g z a = MA\omega^2 \cos \omega t$$

(a) The time dependent driving force is then

$$F(t) = MA\omega^2 \cos \omega t \quad .$$

(b) If we also consider the effects of damping, the equation of motion can be written

$$\ddot{z} + \gamma \dot{z} + \omega_0^2 z = A\omega^2 \cos \omega t$$

Where $\omega_0^2 = \dfrac{\rho g a}{M}$

To determine whether there would be large amplitude oscillations, we can evaluate the expression ω/ω_0 which would approach 1 at resonance.

$$\frac{\omega}{\omega_0} = \frac{2\pi / T}{\sqrt{M/\rho g a}} = \frac{2\pi}{(6\,s)}\sqrt{\frac{(10^3\,kg\cdot m^{-3})(9.81\,m/s^2)(.05\,m^2)}{(100\,kg)}}$$

$$= 2.3$$

The Q-value for this oscillator is

$$Q = \frac{\omega_0}{\gamma} = (10^2\,s)\sqrt{\frac{(100\,kg)}{(10^3\,kg\cdot m^{-3})(9.81\,m/s^2)(.05\,m^2)}}$$

$$= 221$$

Recall that the FWHM of the power as a function of ω/ω_0 is just γ. Therefore, since $\dfrac{\omega/\omega_0}{\gamma} = \dfrac{2.3}{10^{-2}} \gg 1$ the driving frequency is far from resonance so there will be no large amplitude oscillations.

2. The position of the pin on the shaft is

$$z(t) = r \cos \omega t$$

and if the length of the rubber band (minus the equilibrium length) is $x(t)$ then the position of the mass in an inertial reference frame is $z(t) + x(t)$. The equation of motion is then

$$M(\ddot{z} + \ddot{x}) + b\dot{x} + kx = 0$$

or $\quad M\ddot{x} + b\dot{x} + kx = -M\ddot{z} = Mr\omega^2 \cos \omega t$, which can be written in the standard way:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = r\omega^2 \cos \omega t$$

(a) If the motor stopped, the mass would oscillate with $x(t)$ given by

$$x(t) = A e^{-\gamma t/2} \cos(\omega_{free} t)$$

where $\omega_{free} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

If the amplitude of oscillations decreased by $1/e$ in time T then

$$e^{-\gamma T/2} = e^{-1} \Rightarrow \gamma = 2/T.$$

The Q value of the system can be expressed

$$Q = \frac{\omega_0}{\gamma} = \frac{2}{T}\sqrt{\frac{k}{M}}$$

(b) We already worked out the vertical component of the position of the pin. It was

$$z(t) = r\cos\omega t$$

(c) We already worked out the differential equation. It was

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = r\omega^2\cos\omega t$$

(d) The maximal amplitude occurs when $\omega \approx \omega_0$. When $\gamma \ll \omega_0$ this will occur when $\omega \approx \sqrt{k/m}$.

To calculate the exact frequency for which the amplitude is maximal we consider the expression for the amplitude of an oscillator subject to a force $F(t) = F_0\cos\omega t$. This was

$$A(\omega) = \frac{F_0/m}{\left((\omega_0^2-\omega^2)^2 + (\omega\omega_0/Q)^2\right)^{1/2}}$$

In the current problem the force term is of the form $F(t) = mr\omega^2\cos\omega t$ so the amplitude is

$$A(\omega) = \frac{r\omega^2}{\left((\omega_0^2-\omega^2)^2 + (\omega\omega_0/Q)^2\right)^{1/2}}$$

$$= \frac{r\omega^2/\omega_0^2}{\left(\left(1-\omega^2/\omega_0^2\right)^2 + \left(\omega/\omega_0 Q\right)^2\right)^{1/2}}$$

Let $u = \omega^2/\omega_0^2$. $A(u) = \dfrac{ru}{\left((1-u)^2 + u/Q^2\right)^{1/2}}$

$A(u)$ is maximal when $\dfrac{dA}{du} = 0$.

$$\frac{dA}{du} = \frac{r}{((1-u)^2 + u/Q^2)^{1/2}} + \frac{ru(1-u) - ru/2Q^2}{((1-u)^2 + u/Q^2)^{3/2}} = 0$$

The denominator is never zero, so

$$(1-u)^2 + u/Q^2 + u(1-u) - u/2Q^2 = 0$$

$$1 - 2u + \cancel{u^2} + u/Q^2 + u - \cancel{u^2} - u/2Q^2 = 0$$

$$1 - u + u/2Q^2 = 0$$

$$1 - u(1 - 1/2Q^2) = 0$$

$$u = \frac{1}{1 - 1/2Q^2}$$

Therefore, maximal amplitude occurs when

$$\omega = \omega_0 \sqrt{\frac{1}{1 - 1/2Q^2}}$$

This is a higher frequency than ω_0.

aside... $\left\{ \right.$ In the case where the forcing term was

$$F(t) = F_0 \cos \omega t,$$

the maximal amplitude occurs at $\omega = \omega_0 \sqrt{1 - \frac{1}{2Q^2}}$.

In the current problem, when $\omega = \omega_0 \sqrt{\dfrac{1}{1 - \frac{1}{2Q^2}}}$, the maximal amplitude is

$$A_{max} = \frac{r}{\left(1 - \frac{1}{2Q^2}\right)\left(\left(1 - \frac{1}{1-\frac{1}{2Q^2}}\right)^2 + \frac{\frac{1}{Q^2}}{1 - \frac{1}{2Q^2}}\right)^{\frac{1}{2}}}$$

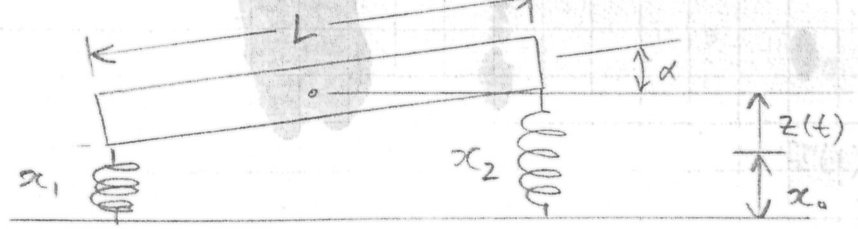
$$= \frac{r}{\left(\left(1 - \frac{1}{2Q^2} - 1\right)^2 + \frac{1}{Q^2}\left(1 - \frac{1}{2Q^2}\right)\right)^{1/2}}$$

$$= \frac{r}{\left(\frac{1}{4Q^4} + \frac{1}{Q^2} - \frac{1}{2Q^4}\right)^{1/2}}$$

$$= \frac{rQ}{\left(1 - \frac{1}{2Q^2}\right)^{1/2}}$$

When Q is large, $A_{max} \approx rQ$.

3.



$2kx_0 = Mg$

$x_0 = \frac{Mg}{2k}$

The deviation of each spring from their equilibrium length is $x_1(t)$ and $x_2(t)$, where

$$x_1(t) = z(t) - L/2\, \alpha(t)$$

$$x_2(t) = z(t) + L/2\, \alpha(t)$$

The net force acting on the beam is

$$F = -kx_1 - kx_2 = M\ddot{z}$$

The net torque on the beam is

$$N = -\frac{kx_1 L}{2} + \frac{kx_2 L}{2} = I\ddot{\alpha}$$

Thus,

(a) $$M\ddot{z} + 2kz = 0$$

$$I\ddot{\alpha} - \frac{k}{2}L^2\alpha = 0$$

(b) The frequencies of the normal modes are

$$\omega_z = \sqrt{\frac{2k}{M}}$$

and

$$\omega_\alpha = \sqrt{\frac{kL^2}{2I}}$$

(c) When one spring is compressed by a distance d, while the other is in its equilibrium position, we have

$$x_1(0) = d$$
$$x_2(0) = 0$$

Hence, $z(0) = \frac{1}{2}(x_1(0) + x_2(0)) = \frac{1}{2}d$

and $\alpha(0) = \frac{1}{L}\left(x_2(0) - x_1(0)\right) = -\frac{d}{L}$

Solutions to the equations of motion are

$$z(t) = A\cos(\omega_z t + \theta)$$
$$\alpha(t) = B\cos(\omega_\alpha t + \varphi)$$

At $t=0$, the velocity is

$$\dot{z}(0) = -A\omega_z \sin\theta$$
$$\dot{\alpha}(0) = -B\omega_\alpha \sin\varphi .$$

If these are both zero at $t=0$, then we must have $\theta = \varphi = 0$.

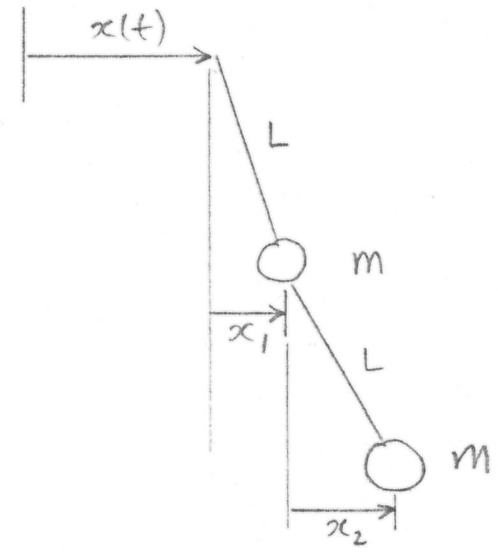
Therefore, $A = \frac{1}{2}d$ and $B = -d/L$.

The solution is

$$z(t) = \frac{d}{2}\cos\omega_z t \quad ; \quad \omega_z = \sqrt{\frac{2k}{M}}$$

$$\alpha(t) = -\frac{d}{L}\cos\omega_\alpha t \quad , \quad \omega_\alpha = \sqrt{\frac{kL^2}{2I}} .$$

4.



$x(t)$ is defined in an inertial reference frame. In this frame we can define the coordinates of the masses:

$$y_1(t) = x(t) + x_1(t)$$
$$y_2(t) = x(t) + x_1(t) + x_2(t) .$$

The tension in each string supports the masses. In the top string, for $x_1 \ll L$,

$$T_1 = 2mg$$

and in the bottom string, for $x_2 \ll L$,

$$T_2 = mg$$

The x-component of the force acting on the top mass is

$$F_1 = -\frac{T_1 x_1}{L} + \frac{T_2 x_2}{L} = -\frac{2mg x_1}{L} + \frac{mg x_2}{L}$$

and the force acting on the bottom mass is

$$F_2 = -\frac{T_2 x_2}{L} = -\frac{mg x_2}{L}$$

These forces give rise to accelerations in the inertial reference frame, and the equations of motion are

$$m\ddot{y}_1 = m\ddot{x} + m\ddot{x}_1 = -\frac{2mg}{L}x_1 + \frac{mg}{L}x_2$$

$$m\ddot{y}_2 = m\ddot{x} + m\ddot{x}_1 + m\ddot{x}_2 = -\frac{mg}{L}x_2$$

$$\Rightarrow \quad \ddot{x}_1 + \frac{2g}{L}x_1 - \frac{g}{L}x_2 = -\ddot{x}$$

$$\ddot{x}_1 + \ddot{x}_2 + \frac{g}{L}x_2 = -\ddot{x}$$

Resonance will occur when ω^2 is an eigenvalue. In this case, suppose

$$x_1(t) = A\cos\omega t$$
$$x_2(t) = B\cos\omega t$$

Then $\ddot{x}_1(t) = -\omega^2 x_1$
$\ddot{x}_2(t) = -\omega^2 x_2$ and the homogeneous system of equations is

$$(-\omega^2 + 2g/L)\,x_1 - g/L\,x_2 = 0$$

$$(-\omega^2)\,x_1 + (-\omega^2 + g/L)\,x_2 = 0$$

or $$\begin{pmatrix} 2g/L - \omega^2 & -g/L \\ -\omega^2 & g/L - \omega^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Thus, $$\begin{vmatrix} 2g/L - \omega^2 & -g/L \\ -\omega^2 & g/L - \omega^2 \end{vmatrix} = 0$$

The determinant is

$$\begin{vmatrix} 2g/L - \omega^2 - \omega^2 & -g/L \\ & g/L - \omega^2 \end{vmatrix} = (2g/L - \omega^2)(g/L - \omega^2) - \omega^2 g/L = 0$$

Let $\lambda = \omega^2$. Then,

$$(2g/L - \lambda)(g/L - \lambda) - \lambda g/L = 0$$

$$\lambda^2 - \frac{4g}{L}\lambda + \frac{2g^2}{L^2} = 0$$

$$\lambda = \frac{2g}{L} \pm \sqrt{\frac{4g^2}{L^2} - \frac{2g^2}{L^2}}$$

$$= \frac{2g}{L} \pm \frac{g\sqrt{2}}{L} = \frac{g}{L}\left(2 \pm \sqrt{2}\right)$$

Resonance will occur when

$$\omega_1 = \sqrt{\frac{g}{L}}\sqrt{2-\sqrt{2}}$$

or

$$\omega_2 = \sqrt{\frac{g}{L}}\sqrt{2+\sqrt{2}}$$

The eigenvectors can be determined by substituting the eigenvalues into the equation:

$$\begin{pmatrix} 2g/L - g/L(2-\sqrt{2}) & -g/L \\ -g/L(2-\sqrt{2}) & g/L - g/L(2-\sqrt{2}) \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = 0$$

$$\begin{pmatrix} g/L\sqrt{2} & -g/L \\ -g/L(2-\sqrt{2}) & -g/L(1-\sqrt{2}) \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = 0$$

$$\Rightarrow \sqrt{2}A_1 - B_1 = 0$$

$$\text{So } A_1 = B_1/\sqrt{2}$$

Likewise

$$\begin{pmatrix} 2g/L - g/L(2+\sqrt{2}) & -g/L \\ -g/L(2+\sqrt{2}) & g/L - g/L(2+\sqrt{2}) \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -g/L\sqrt{2} & -g/L \\ -g/L(2+\sqrt{2}) & -g/L(1+\sqrt{2}) \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = 0$$

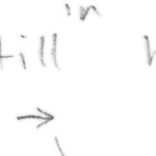
$$\Rightarrow A_2\sqrt{2} + B_2 = 0$$

$$A_2 = -B_2/\sqrt{2}$$

In the first normal mode (lower frequency) both masses more in the same direction, and the top mass has a smaller amplitude

In the second normal mode of oscillation, the masses move in opposite directions but the top mass still has a smaller amplitude

1: $\omega^2 = g/L(2-\sqrt{2})$



2: $\omega^2 = g/L(2+\sqrt{2})$