Assignment # 7

1. The boundary condition $\psi(R,t)=0$ requires that $r=R$ is a node in the function $J_0(kr)$ which we approximate by

$$J_0(z) \sim \sqrt{\frac{2}{\pi}}\ \frac{\cos(z-\pi/4)}{\sqrt{z}}$$

where $z = kr$.

If $r=R$ is a node, then $kR - \pi/4 = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ Thus,

$$k = \frac{\frac{\pi}{2}+\frac{\pi}{4}}{R}, \quad \frac{\frac{3\pi}{2}+\frac{\pi}{4}}{R}, \quad \text{etc.}$$

which we can write as

$$k_n = \frac{(2n-1)\pi/2 + \pi/4}{R}$$
$$= \frac{n\pi}{R} - \frac{\pi}{4R}$$

The frequencies would be

$$\omega_n = v k_n = \frac{\pi v}{R}\left(n - 1/4\right)$$

Although the question did not ask for a comparison, we can compare these approximate roots with the true roots of $J_0(z)$:

n	Rk_n (approx)	z_n (exact)
1	$3\pi/4 = 2.3562$	2.4048
2	$7\pi/4 = 5.4978$	5.5201
3	$11\pi/4 = 8.6394$	8.6537
4	$15\pi/4 = 11.7810$	11.7915
5	$19\pi/4 = 14.9226$	14.9309

not bad!

2. Pressure waves inside the spherical balloon will satisfy the wave equation in spherical coordinates:

$$\nabla^2 \psi = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial \psi}{\partial r} = -\frac{\omega^2}{v^2}\psi$$

provided $\frac{\partial \psi}{\partial \theta} = \frac{\partial \psi}{\partial \phi} = 0$, as we assume.

We can write this as

$$\frac{1}{r}\frac{\partial^2}{\partial r^2} r\psi = -\frac{\omega^2}{v^2}\psi$$

It is useful to check that this is the case:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2} r\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(\psi + r\frac{\partial \psi}{\partial r}\right)$$
$$= \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2}$$
$$= \frac{2}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2}$$

but also,

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial \psi}{\partial r} = \frac{1}{r^2}\left(2r\frac{\partial \psi}{\partial r} + r^2\frac{\partial^2 \psi}{\partial r^2}\right)$$
$$= \frac{2}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2}$$

So now we can comfortably write

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) = -\frac{\omega^2}{v^2}\psi.$$

If we let $\psi(r,t) = \frac{f(r,t)}{r}$ then this is

$$\frac{1}{r}\frac{\partial^2 f}{\partial r^2} = -\frac{\omega^2}{v^2}\frac{f}{r} \Rightarrow \frac{\partial^2 f}{\partial r^2} = -\frac{\omega^2}{v^2} f$$

Solutions are $f(r,t) = A\sin(kr)\cos(\omega t)$ but not $\cos(kr)\cos(\omega t)$ because the function $\psi(r,t) = \frac{f(r,t)}{r}$ must remain finite as $r \to 0$. Since $r = R$ is a node, we must have $kR = n\pi$.

The frequencies of oscillations are then

$$\omega_n = v k_n = \frac{n\pi v}{R}$$

and when we write $v = \sqrt{\frac{\gamma p}{\rho}}$ we have

$$\omega_n = \frac{n\pi}{R}\sqrt{\frac{\gamma p}{\rho}}$$

3.(a) The power carried by a pulse in one transmission line is

$$P = Z I^2$$

If there are equal currents I' propagating in the two transmission lines on the right then the total power is

$$P = 2Z(I')^2 = Z'(2I')^2$$

So the effective impedance is

$$Z' = \frac{Z}{2}.$$

(b) The reflection coefficient is

$$\rho = \frac{Z' - Z}{Z' + Z}$$

$$= \frac{Z/2 - Z}{Z/2 + Z} = \frac{-1/2}{3/2} = -\frac{1}{3} \Rightarrow V_r = -\frac{V_i}{3}$$

The reflected pulse is inverted.

(c) The transmitted pulse has an amplitude

$$V_t = \tau V_i \quad \text{where} \quad \tau = \frac{2Z'}{Z + Z'} = \frac{Z}{Z + Z/2} = \frac{2}{3}$$

The pulses on both transmission lines have this amplitude: $V_t = \dfrac{2V_i}{3}$

Here is another way to reason this out:

The current and voltage in the transmission line are related by

$$V = IZ .$$

The reflected power is then

$$P_r = \frac{V_r^2}{Z} = \frac{\left(-\frac{1}{3}V_i\right)^2}{Z}$$

The transmitted power is then

$$P_t = P_i - P_r = \frac{V_i^2}{Z} - \frac{\left(\frac{1}{3}V_i\right)^2}{Z}$$

$$= \frac{V_i^2}{Z}\left(1 - \frac{1}{9}\right)$$

$$= \frac{8V_i^2}{9Z}$$

Since this power will be split equally on each transmission line, the power on one line is

$$P_t' = \frac{4V_i^2}{9Z} = \frac{\left(\frac{2}{3}V_i\right)^2}{Z}$$

Hence, the voltage on each line is

$$V_t = \frac{2}{3}V_i .$$