Assignment # 6

1. (a) Fourier series representation for the function $y(x) = Ax(L-x)$, $0 < x < L$:

The function can be represented as

$$y(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right)$$

where $A_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) y(x)\,dx$

and $B_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) y(x)\,dx$

Since $y(x) = 0$ when $x = 0$ and $x = L$, we must have $B_n = 0$ for all n.

Since $y(x)$ is symmetric about $x = L/2$, we must have $A_n = 0$ when n is even because $\sin\left(\frac{n\pi x}{L}\right)$ is antisymmetric about $x = L/2$ when n is even.

So, $A_n = \frac{2A}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right) dx$

Change of variables: Let $u = x - L/2$
then $du = dx$

$$A_n = \frac{2A}{L}\int_{-L/2}^{L/2} (L/2 + u)(L/2 - u)\sin\left(\frac{n\pi}{L}(u + L/2)\right) dx$$

$$= \frac{2A}{L}\int_{-L/2}^{L/2} \left[(L/2)^2 - u^2\right] \sin\left(\frac{n\pi u}{L} + \frac{n\pi}{2}\right) du$$

But n must be odd so we will have

$$\sin\left(\frac{n\pi u}{L} + \frac{n\pi}{2}\right) = \pm\cos\left(\frac{n\pi u}{L}\right)$$

where the $(+)$ is for $n = 1, 5, 9, \ldots$
and the $(-)$ is for $n = 3, 7, 11, \ldots$

Thus, $$A_n = \pm\frac{2A}{L}\int_{-L/2}^{L/2}\left[(L/2)^2 - u^2\right]\cos\left(\frac{n\pi u}{L}\right)du$$

$$= \mp\frac{2A}{L}\int_{-L/2}^{L/2}\left[u^2 - (L/2)^2\right]\cos\left(\frac{n\pi u}{L}\right)du$$

Integration by parts: $\displaystyle\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

Let $u' = u^2 - (L/2)^2 \qquad du' = 2u\,du$
$\quad dv' = \cos\left(\frac{n\pi u}{L}\right)du \qquad v' = \frac{L}{n\pi}\sin\left(\frac{n\pi u}{L}\right)$

So $$A_n = \mp\frac{2A}{L}\cancelto{0}{\left[u^2 - (L/2)^2\right]\frac{L}{n\pi}\sin\left(\frac{n\pi u}{L}\right)\Big|_{-L/2}^{L/2}}$$

$$\pm\left(\frac{2A}{L}\right)\left(\frac{2L}{n\pi}\right)\int_{-L/2}^{L/2} u\sin\left(\frac{n\pi u}{L}\right)du$$

Integrate by parts again: Let $u' = u \quad , \quad du' = du$
$dv' = \sin\left(\frac{n\pi u}{L}\right)du$
$v' = \frac{-L}{n\pi}\cos\left(\frac{n\pi u}{L}\right)$

$$A_n = \pm\left(\frac{4A}{n\pi}\right)\left(\frac{-L}{n\pi}\right)\cancelto{0}{u\cos\left(\frac{n\pi u}{L}\right)\Big|_{-L/2}^{L/2}}$$

$$\mp\left(\frac{4A}{n\pi}\right)\left(\frac{-L}{n\pi}\right)\int_{-L/2}^{L/2}\cos\left(\frac{n\pi u}{L}\right)du$$

$$A_n = \pm \frac{4AL}{n^2\pi^2} \int_{-L/2}^{L/2} \cos\left(\frac{n\pi u}{L}\right) du$$

$$= \pm \frac{4AL}{n^2\pi^2} \cdot \frac{L}{n\pi} \sin\left(\frac{n\pi u}{L}\right)\Bigg|_{-L/2}^{L/2}$$

$$= \pm \frac{8AL^2}{n^3\pi^3} \sin\left(\frac{n\pi}{2}\right)$$

$$= \pm \frac{8AL^2}{n^3\pi^3} (\pm 1)$$

where $(+1)$ is for $n = 1, 5, 9, \ldots$
and (-1) is for $n = 3, 7, 11, \ldots$

thus, $A_n = \frac{8AL^2}{n^3\pi^3}$ for $n = 1, 3, 5, 7, \ldots$

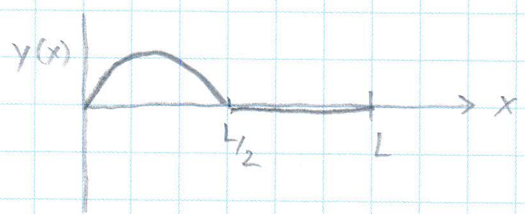
(b) If $y(x) = A \sin\left(\frac{\pi x}{L}\right)$ then the Fourier representation is

$$y(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right)$$

so by inspection $A_1 = A$
and $A_n = 0$, for $n \neq 1$,
while $B_n = 0$ for all n.

(c) When $y(x) = \begin{cases} A \sin(2\pi x/L) & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases}$

the function looks like this:



The coefficients in the Fourier series are

$$A_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) y(x)\,dx$$

$$B_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) y(x)\,dx$$

But since $y(x) = 0$ for $x > L/2$ these are

$$A_n = \frac{2A}{L}\int_0^{L/2} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) dx$$

$$B_n = \frac{2A}{L}\int_0^{L/2} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) dx$$

Using the identities,

$$\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\cos\alpha \sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

the coefficients are

$$A_n = \frac{A}{L}\int_0^{L/2} \cos\left(\frac{(n-2)\pi x}{L}\right)dx - \frac{A}{L}\int_0^{L/2} \cos\left(\frac{(n+2)\pi x}{L}\right)dx$$

$$B_n = \frac{A}{L}\int_0^{L/2} \sin\left(\frac{(n+2)\pi x}{L}\right)dx - \frac{A}{L}\int_0^{L} \sin\left(\frac{(n-2)\pi x}{L}\right)dx$$

We can integrate these directly:

$$A_n = \frac{A}{(n-2)\pi} \sin\left(\frac{(n-2)\pi x}{L}\right)\Bigg|_0^{L/2} - \frac{A}{(n+2)\pi} \sin\left(\frac{(n+2)\pi x}{L}\right)\Bigg|_0^{L/2}$$

$$= \frac{A}{(n-2)\pi} \sin\left(\frac{(n-2)\pi}{2}\right) - \frac{A}{(n+2)\pi} \sin\left(\frac{(n+2)\pi}{2}\right)$$

We might have to worry about the case where $n = 2$ but in this case the first term approaches $\frac{1}{2}A$ while the second term is zero ($\sin 2\pi = 0$).

For other values of n, the coefficients are as follows

$$A_n = \frac{A}{(n-2)\pi} \sin\left(\frac{n\pi}{2} - \pi\right) - \frac{A}{(n+2)\pi} \sin\left(\frac{n\pi}{2} + \pi\right)$$

$$= \frac{-A}{(n-2)\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{A}{(n+2)\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$A_n = \left[\frac{A}{(n+2)\pi} - \frac{A}{(n-2)\pi}\right] \sin\left(\frac{n\pi}{2}\right)$$

$$= \left[\frac{-4A}{\pi(n^2-4)}\right] \sin\left(\frac{n\pi}{2}\right)$$

For even n, this is zero, while for odd values of n, this is

$$A_n = \frac{-4A}{\pi(n^2-4)}(-1)^{(n-1)/2}$$

$$= \frac{4A}{\pi(n^2-4)}(-1)^{(n+1)/2}$$

That is, $A_1 = -\frac{4}{3\pi}A$

$A_2 = \frac{A}{2}$

$A_3 = \frac{4A}{5\pi}$

$A_5 = \frac{-4A}{21\pi}$

$A_6 = \frac{4A}{32\pi}$

etc.

The other coefficients, B_n, must also be zero since none of these terms satisfy the boundary conditions $y(0) = y(L) = 0$.

Note that the solution in the back of the text has the labels A_n and B_n exchanged.

2. (a) The string has initial conditions,

$$y(x,0) = Ax(L-x) \quad \text{and} \quad (\partial y/\partial t)_{t=0} = 0 .$$

The solution is written

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t)$$

where $\omega_n = \dfrac{n\pi v}{L}$.

At $t=0$, $y(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = Ax(L-x)$

and the coefficients are

$$A_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) y(x)\,dx$$

$$= \begin{cases} \dfrac{8AL^2}{n^3\pi^3} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even .} \end{cases} \qquad \left(\text{from problem \#1}\right)$$

The velocity of the string at $t=0$ is

$$\frac{\partial y}{\partial t} = -\sum_{n=1}^{\infty} A_n \omega_n \sin\left(\frac{n\pi x}{L}\right) \sin(\omega_n t)$$

which satisfies the initial condition

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 .$$

(b) When the initial conditions are

$$y(x,0)=0 \quad \text{and} \quad \left(\frac{\partial y}{\partial t}\right)_{t=0} = Bx(L-x)$$

we can represent the function as

$$y(x,t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sin(\omega_n t)$$

where $\omega_n = \dfrac{n\pi v}{L}$

This naturally satisfies $y(x,0)=0$.
The time derivative is

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \omega_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t)$$

and at $t=0$, $\left.\dfrac{\partial y}{\partial t}\right|_{t=0} = \displaystyle\sum_{n=1}^{\infty} B_n \omega_n \sin\left(\frac{n\pi x}{L}\right) = Bx(L-x)$

The coefficients can now be calculated using

$$B_n \omega_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) y(x)\, dx$$

$$= \begin{cases} \dfrac{8BL^2}{\pi^3 n^3} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

Thus, $B_n = \begin{cases} \dfrac{8BL^3}{\pi^4 n^4 v} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$

If this is expressed in terms of the fundamental oscillation frequency,

$$\omega_1 = \frac{\pi v}{L}$$

then the coefficients are

$$B_n = \begin{cases} \dfrac{8BL^2}{\pi^3 n^4 \omega_1} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

3. The speed of wave propogation in the string is

$$v = \sqrt{\frac{T}{\mu}}.$$

The tension is provided by the mass which is assumed to be large compared to the mass of the string.

We know that $\mu L = \frac{M}{100}$ and that $T = Mg$. Thus,

$$\mu = \frac{M}{100\,L}$$

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{Mg \times 100L}{M}}$$
$$= \sqrt{100\,Lg}$$

The time for the pulse to reach the end is $t = \frac{L}{v} = \sqrt{\frac{L}{100\,g}}$

Therefore, $L = 100\,g t^2$

$$= (100)(g)(0.1\,s)^2$$
$$= 9.81\ m.$$

(or $L \approx 10\ m$)

(b) The normal modes of oscillation are

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\omega_n t\right)$$

where $\omega_n = \dfrac{n\pi v}{L} = n\pi\sqrt{\dfrac{100g}{L}}$

But we had previously found that

$$L = g$$ (except with different dimensions)

So it would be better to write

$$L = g \times [s^2]$$

Then the frequency of the third normal mode is

$$\omega_3 = 3\pi\sqrt{100} = 30\pi .$$

The equation is

$$y_3(x,t) = A\sin\left(\frac{3\pi x}{L}\right)\cos\left(30\pi t\right)$$