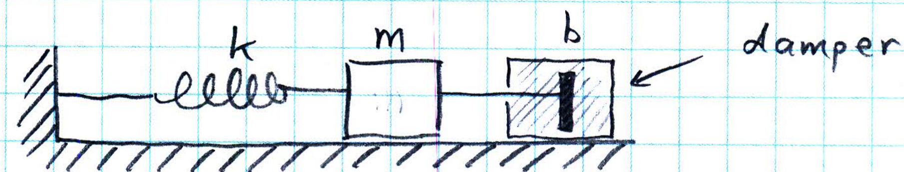
Assignment # 3

1. The physical system looks something like this:



The differential equation that describes the position of the mass is

$$m\ddot{x} + b\dot{x} + kx = 0 .$$

If the mass is displaced by a force equal to mg then the spring is compressed by a distance, h:

$$F = kh = mg$$

Thus, $k = mg/h$.

If the block moves with velocity u, the viscous force is equal to mg.

$$F = bu = mg$$

Thus, $b = mg/u$.

(a) The differential equation is then

$$m\ddot{x} + \frac{mg}{u}\dot{x} + \frac{mg}{h}x = 0$$

or $$\ddot{x} + \frac{g}{u}\dot{x} + \frac{g}{h}x = 0 .$$

(b) Let $x(t) = Ae^{\alpha t}$
Then $\dot{x} = A\alpha e^{\alpha t}$
$\ddot{x} = A\alpha^2 e^{\alpha t}$.

Substituting these into the differential equation gives

$$\left(\alpha^2 + \frac{g}{u}\alpha + \frac{g}{h}\right) x(t) = 0 .$$

Thus, α is a root of the polynomial

$$\alpha^2 + g/u\,\alpha + g/h = 0 .$$

The roots are determined from the quadratic formula:

$$\alpha = -\frac{g}{2u} \pm \sqrt{\frac{g^2}{4u^2} - \frac{g}{h}}$$

If the system oscillates then we must have

$$\frac{g^2}{4u^2} - \frac{g}{h} < 0$$

and the frequency of the damped oscillations is

$$\omega = \sqrt{\frac{g}{h} - \frac{g^2}{4u^2}}$$

when $u = 3\sqrt{gh}$ this is

$$\omega = \sqrt{\frac{g}{h} - \frac{1}{36}\frac{g}{h}} = \sqrt{\frac{g}{h}\,\frac{35}{36}} \approx \sqrt{\frac{g}{h}}$$

(c) The energy is proportional to the square of the displacement.

Thus, at time $t = \frac{u}{g} = \frac{3\sqrt{gh}}{g} = 3\sqrt{\frac{h}{g}}$,

the energy is reduced by a factor of $1/e$.

This is because the position is of the form

$$x(t) = Ae^{-\gamma t/2}\cos(\omega t + \delta)$$

where $\gamma = g/u = \frac{1}{3}\sqrt{\frac{g}{h}}$.

(d) Q is defined by

$$Q = \frac{\omega_0}{\gamma}$$

where ω_0 is the frequency of undamped oscillations.

Thus, $\omega_0 = \sqrt{g/h}$

and $Q = \frac{\sqrt{g/h}}{\frac{1}{3}\sqrt{g/h}} = 3$.

(e) If the oscillator suddenly moves with a finite velocity at $t=0$ then the initial conditions are

$$x(t) = 0$$
$$\dot{x}(t) = p/m$$

where p is the initial momentum.

If $x(t) = A e^{-\gamma t/2} \cos(\omega t - \delta)$

then we must have $\delta = \pi/2$.

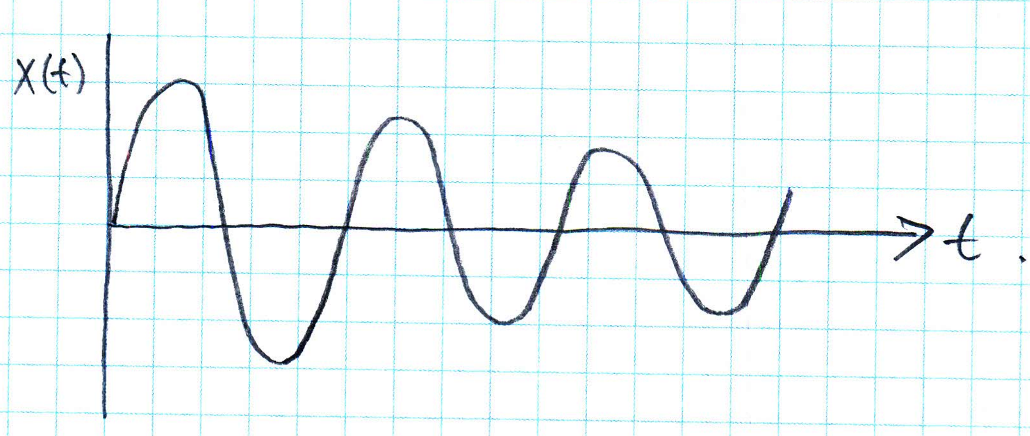
Thus,

$$X(0) = 0$$

and

$$\dot{x}(0) = -A\omega \sin(\omega t - \pi/2)\Big|_{t=0}$$

$$= +A\omega \sin(\pi/2) > 0 .$$



(f) If the oscillator is driven by a force

$$F(t) = mg\cos\omega t$$

where $\omega = \sqrt{2g/h}$, what is the amplitude of the steady state response?

We can use

$$A(\omega) = \frac{F}{k}\frac{\omega_0/\omega}{\left(\left(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2+\frac{1}{Q^2}\right)^{1/2}}$$

where $\omega_0 = \sqrt{g/h}$, $F = mg$, $k = \frac{mg}{h}$

and $Q = 3$

Thus, $\frac{F}{k} = h$, $\frac{\omega_0}{\omega} = \frac{\sqrt{g/h}}{\sqrt{2g/h}} = \frac{1}{\sqrt{2}}$

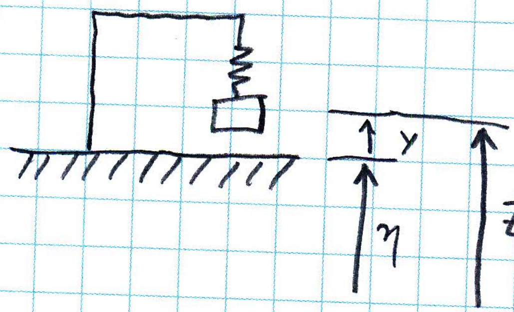
$\frac{1}{Q^2} = \frac{1}{9}$ $\qquad \frac{\omega_0}{\omega} - \frac{\omega}{\omega_0} = \frac{1}{\sqrt{2}} - \sqrt{2} = -\frac{1}{\sqrt{2}}$

Hence, $A(\omega) = \frac{h/\sqrt{2}}{\left(\left(\frac{1}{\sqrt{2}}\right)^2+\frac{1}{9}\right)^{1/2}}$

$$= \frac{h}{\left(1+\frac{2}{9}\right)^{1/2}}$$

$$= h\sqrt{\frac{9}{11}}$$

2. The variables are defined as follows:



where z is in an inertial reference frame then we can write

$$F = -ky = m\ddot{z} = m\ddot{y} + m\ddot{\eta}$$

(a) If we assume some damping mechanism is present, then the differential equation for $y(t)$ is

$$m\ddot{y} + b\dot{y} + ky = -m\ddot{\eta}$$

or
$$\ddot{y} + \gamma\dot{y} + \omega_0^2 y = -\frac{d^2\eta}{dt^2}.$$

(b) If the system undergoes steady state oscillations with amplitude $A(\omega)$ when subjected to a driving force determined by
$$\eta(t) = C\cos\omega t,$$
then
$$\ddot{\eta}(t) = -C\omega^2\cos\omega t$$

$$\ddot{y} + \gamma\dot{y} + \omega_0^2 y = C\omega^2\cos\omega t$$
$$= \frac{F_0}{m}\cos\omega t$$

where $F_0 = mC\omega^2$.

Solutions will be of the form

$$y(t) = A\cos(\omega t - \delta)$$

where
$$A(\omega) = \frac{F_0/m}{\left((\omega_0^2-\omega^2)^2 - \omega^2\gamma^2\right)^{1/2}}$$

$$= \frac{C\omega^2}{\left((\omega_0^2-\omega^2)^2 - \omega^2\gamma^2\right)^{1/2}}$$

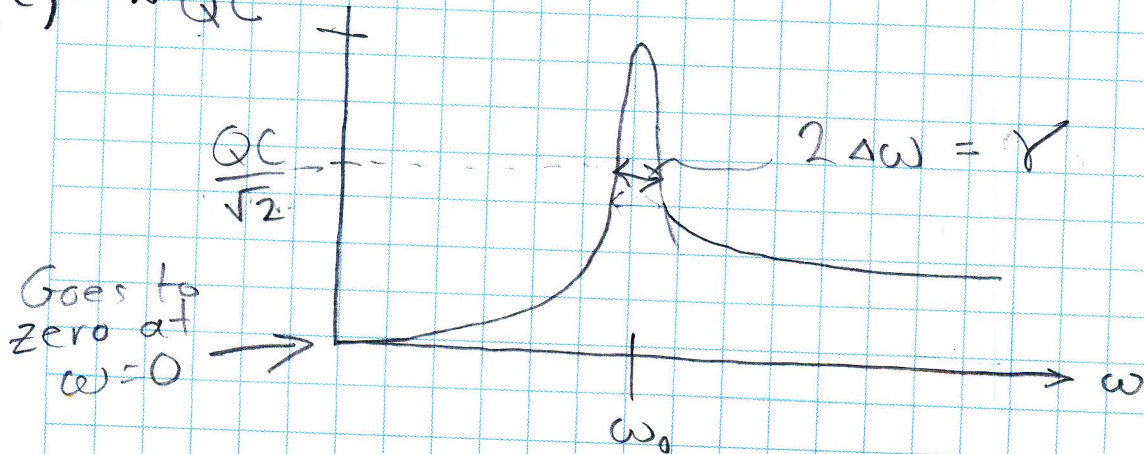
which we can also write

$$A(\omega) = \frac{mC\omega^2}{k} \frac{\omega_0/\omega}{\left(\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right)^{1/2}}$$

$$= \frac{mC\omega_0^2}{k} \frac{\omega/\omega_0}{\left(\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right)^{1/2}},$$

but $\omega_0^2 = k/m$,
$$= C \frac{\omega/\omega_0}{\left(\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right)^{1/2}}$$

(c)



(d) The oscillation frequency is

$$\omega' = \omega_0 \sqrt{1 - \frac{1}{2Q^2}}$$

If $Q = 2$ and $\omega' = \frac{2\pi}{T}$ then

$$\omega_0 = \frac{2\pi}{T\sqrt{1-\frac{1}{2Q^2}}} = \frac{2\pi}{(30\,s)\sqrt{1-\frac{1}{2\times 4}}}$$

$$= \frac{2\pi}{(30\,s)\sqrt{7/8}} = 0.2239\ s^{-1}.$$

If $\omega = \frac{2\pi}{20\ min} = \frac{2\pi}{(20\,min)(60\,s/min)} = 5.236 \times 10^{-3}\ s^{-1}$,

then $\omega/\omega_0 = \frac{5.236 \times 10^{-3}\ s^{-1}}{0.2239\ s^{-1}} = 0.02339$

If the acceleration is $\frac{d^2\eta}{dt^2} = 10^{-9}\ m/s^{-2} = C\omega^2$

then $C = \frac{10^{-9}\ m/s^{-2}}{(5.236 \times 10^{-3} s^{-1})^2} = 36.475 \times 10^{-6}\ m$

Then, $A = C \frac{\omega/\omega_0}{\left(\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right)^{1/2}}$

$$= \frac{(36.48\,\mu m)(0.02339)}{\left(\left(\frac{1}{.02339} - .02339\right)^2 + \frac{1}{4}\right)^{1/2}}$$

$= 0.02\ \mu m.$