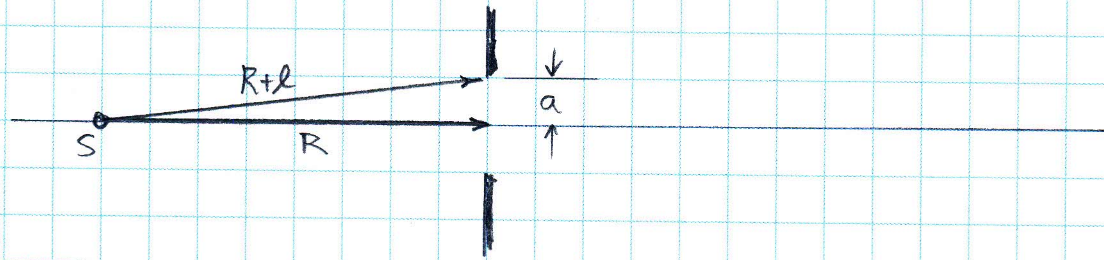
Assignment # 11.

1.



(a)

The path length difference is ℓ where

$$(R+\ell)^2 = R^2 + a^2$$
$$\cancel{R^2} + 2R\ell + \ell^2 = \cancel{R^2} + a^2$$

$$\ell^2 + 2R\ell - a^2 = 0$$

$$\ell = -R + \sqrt{R^2 + a^2}$$
$$= -R + R\sqrt{1 + a^2/R^2}$$
$$\approx -R + R\left(1 + \frac{a^2}{2R^2}\right)$$
$$= \frac{a^2}{2R}$$

The condition for Fraunhofer diffraction was that $\ell \ll \lambda$.

Hence, $\frac{a^2}{2R} \ll \lambda$

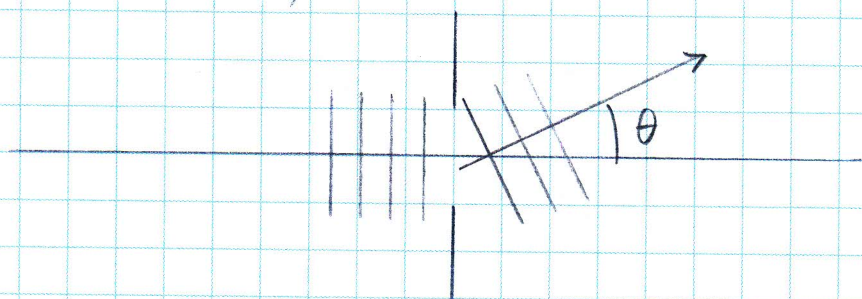
So $\lambda R \gg \frac{a^2}{2}$

(b) When $a = 1$ mm, $\lambda = 500$ nm and $\ell < \lambda/10$, we will have

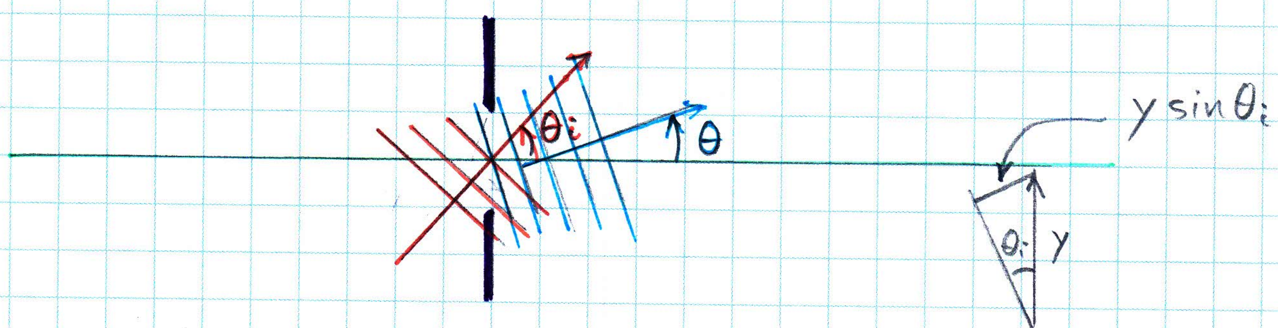
$$R > \frac{a^2}{2\ell} = \frac{a^2}{2\lambda/10} = \frac{10a^2}{2\lambda}$$

$$= \frac{10(10^{-3}\,m)^2}{2(500\times 10^{-9}\,m)} = 10\ m.$$

2. When a plane wave is incident on a slit with $\theta_i = 0$, as shown below, the phase of the wave due to the element of length dy is $\delta(y) = ky\sin\theta$.



When the wave is incident at an angle $\theta_i \neq 0$, as shown,



Now, the phase of the incident wave on the slit is $-ky\sin\theta_i$ so the total phase difference is $\delta(y) = ky(\sin\theta - \sin\theta_i)$. Thus,

$$\beta' = \tfrac{1}{2}kD(\sin\theta - \sin\theta_i).$$

3. The irradiance of a slit of width b is

$$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2$$

where $\beta = \frac{1}{2} k b \sin\theta$.

There are several ways to calculate the limit of $I(\theta)$ as $b \to 0$. One is to recognise that

$$\sin\beta = \beta - \frac{\beta^3}{3!} + \mathcal{O}(\beta^5).$$

so that

$$\frac{\sin\beta}{\beta} = 1 - \frac{\beta^2}{3!} + \mathcal{O}(\beta^4)$$

which approaches 1 as $\beta \to 0$. Hence,

$$\lim_{b\to 0} I(\theta) = \lim_{\beta\to 0} I(0)\left(\frac{\sin\beta}{\beta}\right)$$

$$= \lim_{\beta\to 0} I(0)\left(1 - \frac{\beta^2}{3!} + \mathcal{O}(\beta^4)\right)$$

$$= I(0).$$

Alternatively you can use L'Hopital's rule but you have to differentiate the numerator and denominator twice:

$$\frac{d}{db}\sin^2\beta = 2\sin\beta\cos\beta\,\frac{d\beta}{db} = k\sin\theta\sin\beta\cos\beta$$

$$\frac{d^2}{db^2}\sin^2\beta = k^2\sin^2\theta\left(\cos^2\beta - \sin^2\beta\right) = \frac{1}{2}k^2\sin^2\theta\cos 2\beta$$

$$\frac{d^2}{db^2}(\beta^2) = \frac{1}{2}k^2\sin^2\theta$$

So $$\lim_{b\to 0} I(\theta) = \lim_{b\to 0} I(0)\left(\frac{\frac{1}{2}k^2\sin^2\theta\cos 2\beta}{\frac{1}{2}k^2\sin^2\theta}\right) = I(0)$$

3. (cont.)

In any case, the resulting expression is independent of θ which means that the transmitted light has the same intensity in all directions.

4. The diffraction grating will produce maxima at angles given by

$$\sin\theta = \frac{m\lambda}{a} \qquad \text{where} \qquad a = (10^4\ \text{cm}^{-1})^{-1} = 10^{-6}\ \text{m}.$$

When the diffraction pattern is projected on a screen at a distance d, the position of the line will be

$$y \approx d\sin\theta = \frac{dm\lambda}{a}.$$

(a) The separation between the lines will be

$$\Delta y = \frac{dm}{a}\Delta\lambda = \frac{(1.00\ \text{m})(1)(589.5923 - 588.9953)\times 10^{-9}\ \text{m}}{(10^{-6}\ \text{m})}$$

$$= 0.000597\ \text{m} = 0.597\ \text{mm}.$$

(b) If the lines are just separated, then

$$\Delta\lambda = (\Delta\lambda)_{min} = \frac{\lambda}{\mathcal{R}}$$

where $\mathcal{R}$ is the chromatic resolving power.

Hence, $$\mathcal{R} = \frac{\lambda}{(\Delta\lambda)_{min}} = \frac{\frac{1}{2}(589.5923 + 588.9953)}{(589.5923 - 588.9953)}$$

$$= 987.$$

But $\mathcal{R} = mN$

So $N = \dfrac{\mathcal{R}}{m} = \dfrac{987}{3} = 329$ lines.

This problem seems to overlook the fact that there is no third-order spectrum when $a = 10^{-6}\ m^{-1}$.

If there were, the lines would appear at an angle given by

$$\sin\theta = \frac{3 \cdot (589.3 \times 10^{-9}\ m)}{10^{-6}\ m} \doteq 1.77.$$

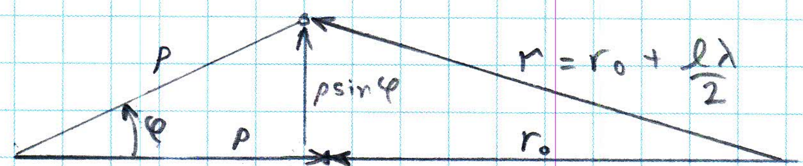
which is impossible. In order to obtain a third-order spectrum we must have instead,

$$a > 3\lambda = 1.77 \times 10^{-6}\ m.$$

5. The element of area for Fresnel zone ℓ is

$$dS = 2\pi \rho^2 \sin\varphi \, d\varphi$$

The geometry used is as follows:



From the law of cosines,

$$r^2 = \rho^2 + (r_0+\rho)^2 - 2\rho(r_0+\rho)\cos\varphi$$

So $$2r\,dr = 2\rho(r_0+\rho)\sin\varphi \, d\varphi$$

$$\rho \sin\varphi \, d\varphi = \frac{r\,dr}{r_0+\rho}$$

So $$dS = \frac{2\pi\rho}{r_0+\rho} r\,dr$$

The limits of integration for zone ℓ are from $r = r_0 + \frac{1}{2}(\ell-1)\lambda$ to $r = r_0 + \frac{1}{2}\ell\lambda$

So $$A_\ell = \frac{2\pi\rho}{r_0+\rho}\int_{r_0+\frac{1}{2}(\ell-1)\lambda}^{r_0+\frac{1}{2}\ell\lambda} r\,dr$$

$$= \frac{\pi\rho}{r_0+\rho}\left[\left(r_0+\tfrac{1}{2}\ell\lambda\right)^2 - \left(r_0+\tfrac{1}{2}(\ell-1)\lambda\right)^2\right]$$

$$= \frac{\pi\rho}{r_0+\rho}\left[\left(r_0+\tfrac{1}{2}\ell\lambda\right)^2 - \left(r_0+\tfrac{1}{2}\ell\lambda\right)^2 + \lambda\left(r_0+\tfrac{1}{2}\ell\lambda\right) - \frac{\lambda^2}{4}\right]$$

$$= \frac{\lambda\pi\rho}{\rho+r_0}\left[r_0 + \tfrac{1}{2}\ell\lambda - \frac{\lambda}{4}\right] = \frac{\lambda\pi\rho}{\rho+r_0}\left[r_0 + \frac{(2\ell-1)\lambda}{4}\right]$$

(b) The mean distance to zone ℓ will be

$$r_\ell = \frac{1}{2}\left(r_0 + (\ell-1)\frac{\lambda}{2} + r_0 + \frac{\ell\lambda}{2} \right)$$

$$= r_0 + \frac{\ell\lambda}{2} - \frac{\lambda}{4}$$

$$= r_0 + \frac{(2\ell-1)\lambda}{4}$$

Thus, the ratio $\frac{A_\ell}{r_\ell} = \frac{\lambda\pi\rho}{\rho + r_0}$

is the same for all zones.