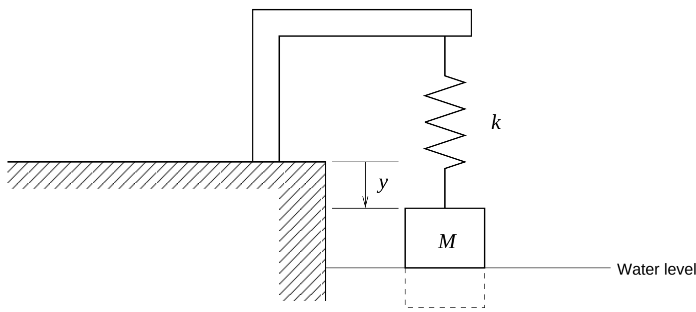
Physics 422 - Spring 2013 - Midterm Exam, March 6$^{\text{th}}$

Answer all questions in the exam booklets provided.
There are 6 questions - please answer all of them.
Explain your reasoning clearly but concisely.
Clearly indicate which work is to be graded.
Each question is of equal weight.
You can use one page of your own notes/formulas.

1. The diagram below shows a cylinder floating in water that is also suspended by a spring with spring constant, k.



The cylinder has a total mass, M, and a cross sectional area A and the water, which has density ρ, provides an upward buoyant force on the cylinder. You can assume that some part of the cylinder always remains under water and that the cylinder is never completely submerged. Suppose that the distance y, as shown on the diagram, has a value of $y = 0$ when the mass is in a state of static equilibrium.

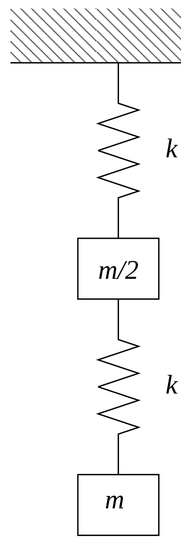
(a) Draw the free-body diagram for the mass indicating the forces that are exterted on it when it has been displaced downward by a distance y.

(b) Write the equations of motion for the mass.

(c) What is the angular frequency of free oscillations?

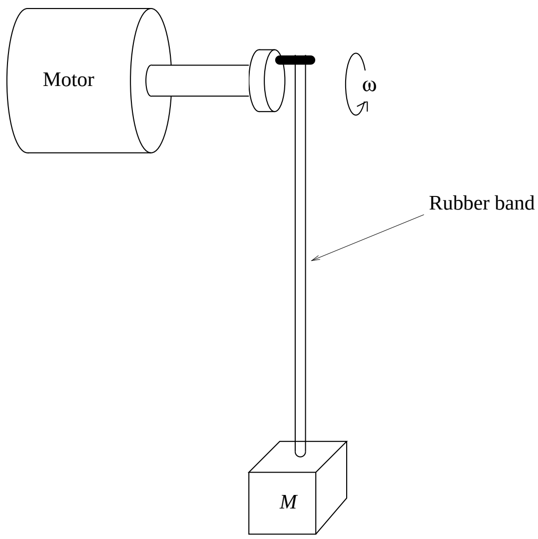
2. A mass, M, is suspended from a rubber band which has an effective spring constant, k. The mass moves in a fluid which produces a damping force with a magnitude equal to $b\dot{y}$, where y represents the vertical positon of the mass.
(a) What is the smallest value of b for which the mass would not oscillate?
(b) If the mass did oscillate, but the amplitude of oscillations decreased by 1/2 in a time T, find an expression for the constant b in terms of M and T.

3. Two different masses are hanging by springs as shown:



(a) Write the equations of motion for the two masses.
(b) Calculate the frequencies of the two normal modes of oscillation in terms of the parameter $\omega_0 = \sqrt{k/m}$.

4. A mass is attached by a rubber band with spring constant k, to a pin on a small wheel that is driven by a motor at an angular frequency ω as shown:



(a) If the pin on the wheel is at a radius r and the mass oscillates in the vertical direction with a maximum amplitude of $10r$ when the motor is driven at a frequency ω_0, what is the value of Q for this oscillating system?
(b) When driven at the angular frequency ω_0, the mass has a peak kinetic energy of $T_{\max}$. By how much should the angular frequency be increased to reduce the peak kinetic energy to the value $T = T_{\max}/2$?

5. A string of length L with linear mass density μ is stretched with tension T between two fixed end points. At time $t = 0$, the displacement and velocity of the string are described by the functions $y(x) = 0$ and $(\partial y/\partial t)_{t=0} = v(x)$ where

$$v(x) = \begin{cases} 2wx/L & x < L/2 \\ 2w(1 - x/L) & x > L/2 \end{cases}$$

which can be expressed as a Fourier series

$$v(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$A_n = \begin{cases} 0 & n = 2, 4, 6, \ldots \\ \pm 8w/n^2\pi^2 & n = 1, 3, 5, \ldots \end{cases}$$

Find an expression for the solution $y(x, t)$ for $t > 0$, expressed in terms of the variables w, L, T and μ.

6. A transparent material has an index of refraction $n(\lambda)$ that is a function of the wavelength of the light propagating through it. Show that the group velocity of light with average wavelength $\bar{\lambda}$ can be written

$$v_g = \frac{c}{n(\bar{\lambda})}\left(1 + \frac{\bar{\lambda}}{n(\bar{\lambda})} n'(\bar{\lambda})\right)$$

in which $n'(\lambda) = dn/d\lambda$.