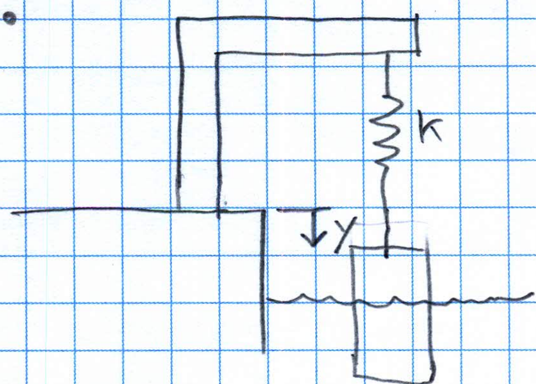
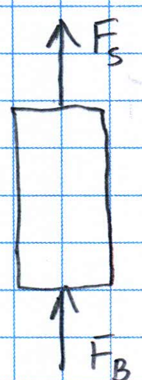
Physics 422 Midterm Exam

1.



When $y = 0$ the cylinder is in equilibrium, so the spring force and the buoyant force balance the force of gravity.

(a) When $y > 0$, corresponding to a displacement in the downward direction, the forces acting on the cylinder are as indicated on the free body diagram



Spring force, $F_s = -ky$

Buoyant force, $F_B = -\rho y A g$

Note that the constant forces due to gravity, the spring and the buoyant force cancel.

(b) Equations of motion:

$$m\ddot{y} = F_s + F_B = -ky - \rho g y A$$

or $$m\ddot{y} + (k + \rho A g)\,y = 0$$

(c) This can be written

$$\ddot{y} + \left(\frac{k + \rho A g}{m}\right) y = \ddot{y} + \omega_0^2 y = 0$$

1(c) Continued,

Angular frequency of free oscillations is

$$\omega_0 = \sqrt{\frac{k + \rho A g}{m}}$$

2. Equation of motion,

$$m\ddot{y} + b\dot{y} + ky = 0$$

We can rewrite this

$$\ddot{y} + \gamma\dot{y} + \omega_0^2 y = 0$$

where $\gamma = b/m$ $\quad \omega_0^2 = \frac{k}{m}$

Solutions should be of the form

$$y(t) = A e^{\alpha t}$$

Thus, $(\alpha^2 + \gamma\alpha + \omega_0^2)\, y(t) = 0$

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

(a) If the mass does not oscillate then it is either overdamped or critically damped. We want the smallest value of b that prevents oscillations, so we have

$$\frac{\gamma^2}{4} = \omega_0^2 \Rightarrow \frac{\gamma}{2} = \omega_0$$

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2(a) Continued,

$$\frac{\gamma}{2} = \omega_0$$

So $\dfrac{b}{2m} = \omega_0$

$$\Rightarrow \quad b = 2m\omega_0 = 2m\sqrt{\frac{k}{m}}$$

$$= 2\sqrt{km}$$

(b) If the amplitude of oscillations decreased by ½ in time T then

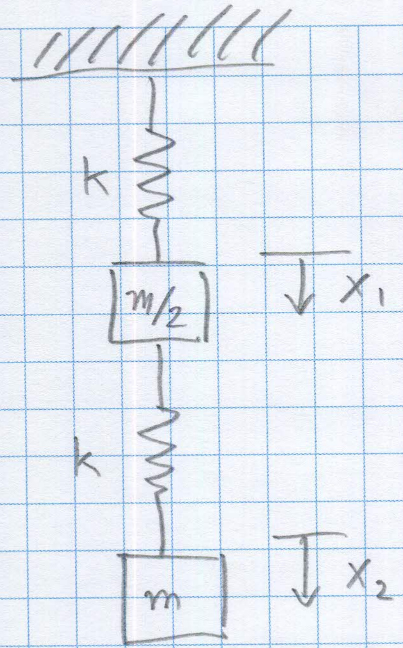
$$e^{-\gamma T/2} = \frac{1}{2}$$

$$\log 2 = \frac{\gamma T}{2}$$

$$\frac{b}{m} = \gamma = \frac{2 \log 2}{T}$$

So $b = \dfrac{2m}{T} \log 2$.

3.



$$\frac{m}{2}\ddot{x}_1 = -kx_1 - k(x_1 - x_2)$$

$$m\ddot{x}_2 = -k(x_2 - x_1)$$

We can write these as

$$\frac{m}{2}\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 + kx_2 - kx_1 = 0$$

or, $$\ddot{x}_1 + 4\omega_0^2 x_1 - 2\omega_0^2 x_2 = 0$$
$$\ddot{x}_2 + \omega_0^2 x_2 - \omega_0^2 x_1 = 0$$

where $\omega_0^2 = \frac{k}{m}$

3(b) If we assume that solutions are of the form

$$x_i(t) = A_i \cos \omega t$$

then the equations of motion are

$$-\omega^2 x_1 + 4\omega_0^2 x_1 - 2\omega_0^2 x_2 = 0$$

$$-\omega^2 x_2 + \omega_0^2 x_2 - \omega_0^2 x_1 = 0$$

We write this as a matrix:

$$\begin{pmatrix} -\omega^2 + 4\omega_0^2 & -2\omega_0^2 \\ -\omega_0^2 & -\omega^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

thus, $\det \begin{pmatrix} \lambda - 4\omega_0^2 & 2\omega_0^2 \\ \omega_0^2 & \lambda - \omega_0^2 \end{pmatrix} = 0$

where $\lambda = \omega^2$.

Therefore, $(\lambda - 4\omega_0^2)(\lambda - \omega_0^2) - 2\omega_0^4 = 0$

$$\lambda^2 - 5\omega_0^2 + 4\omega_0^4 - 2\omega_0^4 = 0$$

$$\lambda^2 - 5\omega_0^2 + 2\omega_0^4 = 0$$

$$\lambda = \omega_0^2 \left(\frac{5 \pm \sqrt{25-8}}{2} \right)$$

$$= \omega_0^2 \left(\frac{5 \pm \sqrt{17}}{2} \right)$$

Frequencies of the normal modes of oscillation are

$$\omega = \omega_0 \sqrt{\frac{5 \pm \sqrt{17}}{2}}$$

4. (a) The relation between the amplitude of the driving motion and the amplitude of oscillations at the resonant frequency is

$$\frac{10r}{r} = Q$$

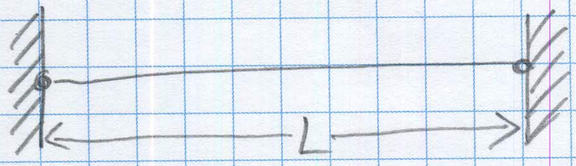
Therefore, $Q = 10$.

(b) We defined $Q = \frac{\omega_0}{\gamma}$

and increasing the frequency from ω_0 to $\omega_0 + \gamma/2$ would decrease the energy by $1/2$.

Hence, $\Delta\omega = \frac{\gamma}{2} = \frac{\omega_0}{2Q} = \frac{\omega_0}{20}$

5.



Initial position is $y(x) = 0$

Initial velocity is $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v(x)$

where $v(x) = \begin{cases} 2wx/L & \text{for } x < L/2 \\ 2w(1 - x/L) & \text{for } x > L/2 \end{cases}$

$$= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

where $A_n = \begin{cases} 0 & n = 2, 4, 6 \\ \pm \frac{8w}{n^2\pi^2} & n = 1, 3, 5, \ldots \end{cases}$

Let us suppose that $y(t)$ is of the form

$$y(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sin(\omega_n t)$$

where $\omega_n = k_n v = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$

This naturally satisfies the initial condition

$$y(0) = 0 .$$

5. (continued)

The first derivative is

$$\dot{y}(t) = \sum_{n=1}^{\infty} B_n \omega_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t)$$

So at $t=0$, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} B_n \omega_n \sin\left(\frac{n\pi x}{L}\right)$

But this is just $v(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$

So $B_n = \frac{A_n}{\omega_n} = \begin{cases} 0 & \text{for } n = 2, 4, 6 \\ \pm \frac{8wL}{n^3 \pi^3} \sqrt{\frac{\mu}{T}} & n = 1, 3, 5, \cdots \end{cases}$

6. Group velocity is

$$v_g = \frac{d\omega}{dk} \quad \text{where } \omega = vk = \frac{c}{n} k$$

Thus, $v_g = \frac{d\omega}{dk} = \frac{c}{n} - \frac{ck}{n^2} \frac{dn}{dk}$

$$= \frac{c}{n}\left(1 - \frac{k}{n} \frac{dn}{d\lambda} \frac{d\lambda}{dk}\right)$$

But $\lambda = \frac{2\pi}{k}$ so $\frac{d\lambda}{dk} = -\frac{2\pi}{k^2}$

$$v_g = \frac{c}{n}\left(1 + \frac{2\pi}{nk} \frac{dn}{d\lambda}\right)$$

$$= \frac{c}{n}\left(1 + \frac{\lambda}{n(\lambda)} \frac{dn}{d\lambda}\right)$$

6 (continued)

We would evaluate the group velocity at the average wavelength of a wave packet, so we would write

$$v_g = \frac{c}{n(\bar{\lambda})}\left(1 + \frac{\bar{\lambda}}{n(\bar{\lambda})}\, n'(\bar{\lambda})\right)$$