Problem set 6, Due Mar 8

March 7, 2011

- An infinitely long cylinder of plasma, with the radius R, carries current with the uniform current density $J = J_z$ along the axis. Find the pressure distribution required for equilibrium.

$$
\begin{aligned}
& I = \pi r^2 J_z \\
& B_\phi = \frac{2I}{rc} = 2\pi r J_z \\
& \vec{J} \times \mathbf{B} = -2\pi r J_z^2 = \nabla P = \partial_r P \\
& P = P_0 - \pi r^2 J_z^2
\end{aligned} \tag{1}
$$

- Magnetic field is given as $B = B_0 \tanh(x/L)\hat{y}$. Find the current and density distribution if $p = C\rho^\gamma$

$$
\begin{aligned}
& \jmath = \frac{c}{4\pi} \nabla \times \mathbf{B} = \frac{c}{4\pi} \frac{B_0}{L} \cosh^{-2}(x/L) \mathbf{e}_z \\
& \jmath \times \mathbf{B} = -\frac{c}{4\pi} \frac{B_0^2}{L} \tanh(x/L) \cosh^{-2}(x/L) \mathbf{e}_x = \partial_x P \\
& P = (B_0^2/2) \frac{c}{4\pi} \cosh^{-2}(x/L) \\
& \rho \propto P^{1/\gamma} = \left((B_0^2/2) \frac{c}{4\pi} \cosh^{-2}(x/L) \right)^{1/\gamma}
\end{aligned} \tag{2}
$$

- Plasma is confined in a long axisymmetric cylinder of radius R. Plasma is strongly magnetized, $\beta \ll 1$ (so it relaxes to a force-free equilibrium). Total axial current (along the axis of the cylinder) is zero. Assuming that the force-free equilibrium is the "constant α", find α.

  Force-free Grad-Shafranov equation in cylindrical coordinates

$$
r\partial_r \left((1/r)\partial_r \Psi \right) + \partial_z^2 \Psi + 16\pi^2 I I' = 0 \tag{3}
$$

  Assume independent of z and linear $I = \alpha/(4\pi)\Psi$.

$$
\begin{aligned}
& r\partial_r \left((1/r)\partial_r \Psi \right) + \alpha^2 \Psi = 0 \\
& \Psi = r J_1(\alpha r) \frac{B_0}{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
B_z &= B_0 J_0(\alpha r) \\
B_\phi &= \frac{2I}{r} = \alpha/(2\pi)\frac{\Psi}{r} = \frac{1}{2\pi} J_1(\alpha r) B_0
\end{aligned} \tag{4}
$$

Condition $B_\phi(R) = 0$ requires $J_1(\alpha R) = 0$, $R = 3.83/\alpha$.