Physics 565 - Spring 2011, Assignment #4, Due Feburary 25$^{\text{th}}$

1. Suppose a spin-1/2 field, $\psi(x)$ is coupled to a *classical* field $A^\mu(x)$. In this sense, the presence of $\psi(x)$ does not change the field $A^\mu(x)$, which can then be regarded as just a 4-vector function function of x.

(a) Using the minimal substitution prescription, $i\partial^\mu \to i\partial^\mu + eA^\mu$, show how the free Lagrangian density $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ is modified by the presence of A^μ and use determine the equations of motion.

(b) Find an expression for the Hamiltonian, that is,

$$H = \int d^3x T^{00}(x),$$

expressed in terms of the field $\psi(x)$ that is assumed to satisfy the free-field equations of motion.

(c) Suppose that $A^\mu(x) = (\Phi(x), \vec{0})$ is an electromagnetic potential representing a static electric field. Find an expression for the Hamiltonian in terms of the number operators for the spin-1/2 field. Use the representation of the Dirac fields,

$$\begin{aligned}
\psi(x) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\sum_{\lambda=1,2}\left[u^{(\lambda)}(k)b_\lambda(k)e^{-ik\cdot x} + v^{(\lambda)}(k)d^\dagger_\lambda(k)e^{ik\cdot x}\right] \\
\bar{\psi}(x) &= \int \frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega_{k'}}\sum_{\lambda'=1,2}\left[\bar{u}^{(\lambda')}(k')b^\dagger_{\lambda'}(k')e^{ik'\cdot x} + \bar{v}^{(\lambda')}(k')d_{\lambda'}(k')e^{-ik'\cdot x}\right]
\end{aligned}$$

where the creation and annihilation operators satisfy

$$\begin{aligned}
\left\{d^\dagger_\lambda(k), d_{\lambda'}(k')\right\} &= (2\pi)^3 2\omega_k \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}') \\
\left\{b^\dagger_\lambda(k), b_{\lambda'}(k')\right\} &= (2\pi)^3 2\omega_k \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}') \\
\left\{d^\dagger_\lambda(k), d_{\lambda'}(k)\right\} &= \delta_{\lambda\lambda'} \\
\left\{b^\dagger_\lambda(k), b_{\lambda'}(k)\right\} &= \delta_{\lambda\lambda'}
\end{aligned}$$

and the spinors are normalized so that

$$\begin{aligned}
u^\dagger_\lambda(k)u_{\lambda'}(k) &= 2\omega_k\delta_{\lambda\lambda'} \\
v^\dagger_\lambda(k)v_{\lambda'}(k) &= 2\omega_k\delta_{\lambda\lambda'} \\
u^\dagger_\lambda(k)v_{\lambda'}(k) &= 0 \\
v^\dagger_\lambda(k)u_{\lambda'}(k) &= 0
\end{aligned}$$

2. In the chiral representation, the matrix $\frac{1}{2}\vec{\Sigma}\cdot\hat{k}$ can be written

$$\frac{1}{2}\vec{\Sigma}\cdot\hat{k} = \frac{1}{2|\vec{k}|}\begin{pmatrix} \vec{\sigma}\cdot\vec{k} & 0 \\ 0 & \vec{\sigma}\cdot\vec{k} \end{pmatrix}. \tag{1}$$

(a) Using the chiral representation, show that

$$\frac{1}{2}\vec{\Sigma}\cdot\hat{k}u(k) = \frac{1}{2}\gamma^5 u(k) \tag{2}$$

in the limit $E \gg m$, where $u(k)$ can be expressed in terms of a particle at rest using

$$u(k) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix} E+m+\vec{\sigma}\cdot\vec{k} & 0 \\ 0 & E+m-\vec{\sigma}\cdot\vec{k} \end{pmatrix} u(0)$$

(b) Show that the projection operators

$$P_{\pm} \;=\; \frac{1\pm\gamma^5}{2}$$

select only the components of $\psi(x)$ that transform under Lorentz transformations as $e^{\pm\vec{\sigma}/2\cdot\vec{\phi}}$.
(c) If $\frac{1}{2}\vec{\Sigma}\cdot\hat{k}$ is interpreted as an operator that gives the projection of the spin along an axis that points in the direction of the momentum vector, show that $P_+\psi(x)$ represents a field with positive helicity and that $P_-\psi(x)$ represents a field with negative helicity.