1. Show that the normalization of states:
\[ \langle k | k' \rangle = (2\pi)^3 \cdot 2E \cdot \delta^3(\vec{k} - \vec{k}') \]
is invariant under a Lorentz boost in the x-direction.

2. Show that the Lorentz invariant measure
\[ \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2E} \]
can be written in the manifestly Lorentz covariant form:
\[ \frac{d^4k}{(2\pi)^4} \cdot (2\pi)\delta(k^2 - m^2)\theta(k^0) \]
where
\[ \theta(x) = \begin{cases} 
0 & \text{if } x < 0 \\
1 & \text{if } x > 0 
\end{cases} \]

3. A contravariant tensor has an upper Lorentz index and transforms under a Lorentz transformation like
\[ x'^\mu = \Lambda^\mu_{\nu} x^\nu. \]
Therefore, the Lorentz transformation matrix can be expressed
\[ \Lambda^\mu_{\nu} = \frac{\partial x'^\mu}{\partial x^\nu} \]
a. Show that the partial derivative operator,
\[ \partial_\mu = \frac{\partial}{\partial x^\mu} \]
does not change under Lorentz transformations in this way, but instead transforms like
\[ \partial'_\mu = (\Lambda^{-1})^\nu_{\mu} \partial_\nu \]
Tensors that transform in this way are called covariant tensors, and have a lower Lorentz index.

b. Illustrate by means of an example using a Lorentz transformation in the x-direction only, that
\[ (\Lambda^{-1})^\nu_{\mu} = g_{\mu\sigma}g^{\nu\rho}\Lambda^\sigma_{\rho} \]
c. Show that the components of the momentum operator, \( P^\mu \) can be represented in coordinate space by
\[ P^\mu = i\hbar \partial^\mu \]
4. The 4-vectors for the electromagnetic potential and the electric current are

\[ A^\mu = (\Phi, A) \]
\[ j^\mu = (\rho, j) \]

where \( \Phi(x) \) is the electromagnetic scalar potential, \( A(x) \) is the electromagnetic vector potential, \( \rho(x) \) is the charge density and \( j(x) \) is the current density. Show that Maxwell’s equations

\[ \nabla \cdot E = \rho \]
\[ \nabla \times B - \frac{\partial E}{\partial t} = j \]
\[ \nabla \cdot B = 0 \]
\[ \nabla \times E + \frac{\partial B}{\partial t} = 0 \]

can be written

\[ \partial_\mu F^{\mu\nu} = j^\nu \]

where

\[ F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]

Recall that

\[ E = -\nabla \Phi - \frac{\partial A}{\partial t} \]
\[ B = \nabla \times A. \]