Physics 565 - Fall 2011, Assignment #1, Due January 21$^{\text{nd}}$

1. Show that the normalization of states:

$$\langle k|k'\rangle \;=\; (2\pi)^3 \cdot 2E \cdot \delta^3(\vec{k} - \vec{k}')$$

is invariant under a Lorentz boost in the x-direction.
2. Show that the Lorentz invariant measure

$$\frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2E}$$

can be written in the mainifestly Lorentz covariant form:

$$\frac{d^4k}{(2\pi)^4} \cdot (2\pi)\delta(k^2 - m^2)\theta(k^0)$$

where

$$\theta(x) \;=\; \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

3. A *contravariant* tensor has an upper Lorentz index and transforms under a Lorentz transformation like

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}.$$

Therefore, the Lorentz transformation matrix can be expressed

$$\Lambda^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}$$

a. Show that the partial derivative operator,

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$$

does not change under Lorentz transformations in this way, but instead transforms like

$$\partial'_{\mu} = (\Lambda^{-1})_{\mu}{}^{\nu}\partial_{\nu}$$

Tensors that transform in this way are called *covariant* tensors, and have a lower Lorentz index.
b. Illustrate by means of an example using a Lorentz transformation in the x-direction only, that

$$(\Lambda^{-1})_{\mu}{}^{\nu} = g_{\mu\sigma} g^{\nu\rho} \Lambda^{\sigma}{}_{\rho}$$

c. Show that the components of the momentum operator, P^{μ} can be represented in coordinate space by

$$P^{\mu} = i\hbar\partial^{\mu}$$

4. The 4-vectors for the electromagnetic potential and the electric current are

$$\begin{aligned} A^\mu &= (\Phi, \boldsymbol{A}) \\ j^\mu &= (\rho, \boldsymbol{j}) \end{aligned}$$

where $\Phi(x)$ is the electromagnetic scalar potential, $\boldsymbol{A}(x)$ is the electromagnetic vector potential, $\rho(x)$ is the charge density and $\boldsymbol{j}(x)$ is the current density. Show that Maxwell's equations

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= \rho \\ \nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} &= \boldsymbol{j} \\ \nabla \cdot \boldsymbol{B} &= 0 \\ \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} &= 0 \end{aligned}$$

can be written

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= j^\nu \\ \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} &= 0 \end{aligned}$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

Recall that

$$\begin{aligned} \boldsymbol{E} &= -\nabla\Phi - \frac{\partial \boldsymbol{A}}{\partial t} \\ \boldsymbol{B} &= \nabla \times \boldsymbol{A}. \end{aligned}$$