1. \( \mathcal{L} = (D^\mu \phi) \cdot (D_\mu \phi) - m^2 \phi \cdot \phi - \frac{1}{4} F_{\mu \nu} \cdot F^{\mu \nu} \)

where \( D^\mu \phi = \partial^\mu \phi + g \, W^\mu \times \phi \)
\( F_{\mu \nu} = \partial_{\mu} W_{\nu} - \partial_{\nu} W_{\mu} + g \, W_{\mu} \times W_{\nu} \)

(a) Under a gauge transformation,
\( \tilde{W}_\mu \rightarrow \tilde{W}'_\mu = \tilde{W}_\mu - \tilde{\lambda}(x) \times \tilde{W}_\mu + \frac{1}{g} \partial_\mu \tilde{\lambda}(x) \)

Consider the terms in the field tensor separately:
\( \partial_\mu \tilde{W}_\nu \rightarrow \partial'_\mu \tilde{W}'_\nu = \partial_\mu \tilde{W}_\nu - (\partial_\mu \tilde{\lambda}(x)) \times \tilde{W}_\nu - \tilde{\lambda}(x) \times (\partial_\mu \tilde{W}_\nu) + \frac{1}{g} \partial_\mu \partial_\nu \tilde{\lambda}(x) \)

\( g \tilde{W}_\mu \times \tilde{W}_\nu \rightarrow g \tilde{W}'_\mu \times \tilde{W}'_\nu = g \tilde{W}_\mu \times \tilde{W}_\nu - g(\partial_\mu \tilde{\lambda}(x) \times \tilde{W}_\nu) \times \tilde{W}_\nu 
- g \tilde{W}_\mu \times (\tilde{\lambda}(x) \times \tilde{W}_\nu) + (\partial_\mu \tilde{\lambda}(x)) \times \tilde{W}_\nu 
+ \tilde{W}_\mu \times (\partial_\nu \tilde{\lambda}(x)) + \mathcal{O}(\lambda^2) \)

But recall that \((\tilde{A} \times \tilde{B}) \times \tilde{C} + \tilde{B} \times (\tilde{A} \times \tilde{C}) = \tilde{A} \times (\tilde{B} \times \tilde{C})\)
so this can be written:
\( g \tilde{W}_\mu \times \tilde{W}_\nu = g \tilde{W}_\mu \times \tilde{W}_\nu - g \tilde{\lambda}(x) \times (\tilde{W}_\mu \times \tilde{W}_\nu) \)
\( + (\partial_\mu \tilde{\lambda}(x)) \times \tilde{W}_\nu + \tilde{W}_\mu \times (\partial_\nu \tilde{\lambda}(x)) + \mathcal{O}(\lambda^2) \)

Hence, \( F_{\mu \nu} \rightarrow F'_{\mu \nu} = \partial_\mu \tilde{W}'_\nu - \partial_\nu \tilde{W}'_\mu + g \tilde{W}'_\mu \times \tilde{W}'_\nu \)
\( = \partial_\mu \tilde{W}_\nu - \partial_\nu \tilde{W}_\mu - (\partial_\mu \tilde{\lambda}(x)) \times \tilde{W}_\nu - \tilde{\lambda}(x) \times (\partial_\mu \tilde{W}_\nu) + \frac{1}{g} \partial_\mu \partial_\nu \tilde{\lambda}(x) 
+ g \tilde{W}_\mu \times \tilde{W}_\nu - g \tilde{\lambda}(x) \times (\tilde{W}_\mu \times \tilde{W}_\nu) 
+ (\partial_\mu \tilde{\lambda}(x)) \times \tilde{W}_\nu + \tilde{W}_\mu \times (\partial_\nu \tilde{\lambda}(x)) + \mathcal{O}(\lambda^2) \)
Therefore, \( F_{\mu \nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + g W_\mu \times W_\nu \)
\[ - \lambda \times ( \partial_\mu W_\nu - \partial_\nu W_\mu + g W_\mu \times W_\nu ) + \text{C}(t). \]

So the field tensor \( F_{\mu \nu} \) transforms in the same way that the fields \( \phi \) did:
\[ \phi \rightarrow \phi' = \phi - \lambda \times \phi. \]
\[ F_{\mu \nu} \rightarrow F'_{\mu \nu} = F_{\mu \nu} - \lambda \times F_{\mu \nu}. \]

We know already that \( \phi \cdot \phi \) is invariant and so for the same reason, \( F_{\mu \nu} \cdot F_{\mu \nu} \) must also be invariant.

Therefore, the term \( \frac{1}{4} F_{\mu \nu} \cdot F_{\mu \nu} \) and hence, the entire Lagrangian must be gauge invariant.
\[ L = (D^\mu \bar{\phi}) \cdot (D_\mu \phi) - m^2 \bar{\phi} \cdot \phi - \frac{i}{4} F_{\mu\nu} \cdot \overline{F_{\mu\nu}} \]

\[ D_\mu \bar{\phi} = \partial_\mu \bar{\phi} + g A_\mu \times \bar{\phi} \]

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g A_\mu \times A_\nu \]

**Lowest order terms:**

\[ L_0 = (\partial_\mu \bar{\phi}) \cdot (\partial^\mu \phi) - m^2 \bar{\phi} \cdot \phi - \frac{i}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) \cdot (\partial^\mu \overline{A^\nu} - \partial^\nu \overline{A^\mu}) \]

**Terms that are first-order in \( g \):**

\[ L_g = g (\partial_\mu \bar{\phi}) \cdot (\overline{A^\mu} \times \bar{\phi}) + g (A_\mu \times \bar{\phi}) \cdot (\partial^\mu \phi) \]

\[ - \frac{g}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) \cdot (\overline{A^\mu} \times A_\nu) \]

\[ - \frac{g}{4} (\overline{A^\mu} \times \overline{A^\nu}) \cdot (\partial_\mu A_\nu - \partial_\nu A_\mu) \]

\[ = 2g (\partial_\mu \bar{\phi}) \cdot (\overline{A^\mu} \times \bar{\phi}) - \frac{g}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) \cdot (A_\mu \times A_\nu) \]

**Terms of second order in \( g \):**

\[ L_{g^2} = g^2 (\overline{A^\mu} \times \bar{\phi}) \cdot (A_\mu \times \phi) + g^2 (\overline{A^\mu} \times A_\nu) \cdot (\overline{A^\mu} \times \overline{A^\nu}) \]

Notice that terms of order \( g \) have 3 fields while terms of order \( g^2 \) have 4 fields. These will correspond to interactions of the form:

- \((\partial_\mu \phi) \cdot (\overline{A^\mu} \times \bar{\phi})\)
- \((\partial_\mu \overline{A^\mu}) \times (\overline{A^\mu} \times \bar{\phi})\)
- \((\overline{A^\mu} \bar{\phi}) \cdot (\overline{A^\mu} \times \bar{\phi})\)
- \((\overline{A^\mu} \times \overline{A^\mu}) \cdot (\overline{A^\mu} \times \overline{A^\mu})\)
(c) Consider the term
\[ L_m = m^2 \mathbf{W}_\mu \cdot \mathbf{W}^\mu \]

Under a gauge transformation,
\[ \mathbf{W}_\mu \rightarrow \mathbf{W}_\mu' = \mathbf{W}_\mu - \mathbf{\lambda}(x) \times \mathbf{W}_\mu + \frac{1}{g} \mathbf{\epsilon}_\mu \mathbf{\lambda}(x) \]

So to first order in \( \mathbf{\lambda} \):
\[ L'_m = m^2 \mathbf{W}_\mu' \cdot \mathbf{W}^\mu' - m^2 (\mathbf{\lambda} \times \mathbf{W}_\mu) \cdot \mathbf{W}^\mu - m^2 \mathbf{W}_\mu \cdot (\mathbf{\lambda} \times \mathbf{W}^\mu) \]
\[ + \frac{m^2}{g} \mathbf{W}_\mu \cdot (\mathbf{\epsilon}_\mu \mathbf{\lambda}) + \frac{m^2}{g} (\mathbf{\epsilon}_\mu \mathbf{\lambda}) \cdot \mathbf{W}^\mu + O(\mathbf{\lambda}^2) \]

Two of the terms vanish because \( \mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0 \) but we are still left with
\[ L'_m = m^2 \mathbf{W}_\mu \cdot \mathbf{W}^\mu + \frac{2m^2}{g} \mathbf{\epsilon}_\mu \mathbf{\lambda} \cdot \mathbf{W}_\mu \]

there are no other terms of order \( \mathbf{\lambda} \) so this expression cannot be gauge invariant unless \( m^2 = 0 \).