Physics 565 - Pauli and Dirac Matrices

The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

and they satisfy the following identity:

$$\sigma^i \sigma^j = i\epsilon_{ijk}\sigma^k + \delta_{ij} \tag{4}$$

in which

$$\epsilon^{ijk} = \epsilon_{ijk} = \begin{cases} 1 & \text{for even permutations of } (ijk) \\ -1 & \text{for odd permutations of } (ijk) \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

It follows that

$$\left[\sigma^i, \sigma^j\right] = 2i\epsilon_{ijk}\sigma^k \tag{6}$$

and

$$\begin{aligned} (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) &= (\sigma^i a^i)(\sigma^j b^j) && (7) \\ &= i a^i b^j \epsilon_{ijk}\sigma^k + a^i b^j \delta_{ij} && (8) \\ &= i(\vec{a}\times\vec{b})\cdot\vec{\sigma} + \vec{a}\cdot\vec{b} && (9) \\ &= i(\vec{\sigma}\times\vec{a})\cdot\vec{b} + \vec{a}\cdot\vec{b}. && (10) \end{aligned}$$

We will use the following two representations of the Dirac matrices.
In the *chiral* representation,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{11}$$

$$\vec{\gamma} = \begin{pmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \tag{12}$$

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13}$$

while in the *standard*, or *Dirac* representation,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (14)$$

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad (15)$$

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (16)$$

In all representations, these matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \quad (17)$$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \quad (18)$$

$$\gamma_5^\dagger = \gamma_5 \quad (19)$$

$$\gamma_5^2 = 1 \quad (20)$$

$$\{\gamma_5, \gamma^\mu\} = 0 \quad (21)$$

The tensor $\sigma^{\mu\nu}$ is defined thus:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \quad (22)$$

and the matrix $\vec{\Sigma}$ is defined

$$\vec{\Sigma} = \gamma_5\gamma^0\vec{\gamma} \quad (23)$$

or

$$\Sigma^i = \frac{1}{2}\epsilon_{ijk}\sigma^{jk} \quad (24)$$

which can be expressed

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \quad (25)$$

in both the chiral and the standard representations.