Physics 565 - Fall 2010, Assignment #3, Due Feburary 17$^{\text{th}}$

1. Consider the current operator $j^\mu(x) = \bar{\psi}\gamma^\mu\psi$ which is constructed from Dirac field operators $\bar{\psi}(x)$ and $\psi(x)$.

(a) Show that $\partial_\mu j^\mu(x) = 0$ and hence, that $j^\mu(x)$ is a conserved current with an associated charge $Q = \int d^3x j^0(x)$.
(b) Using the representation of the Dirac fields,

$$\begin{aligned} \psi(x) &= \int \frac{d^3k}{(2\pi)^3}\frac{m}{\omega_k}\sum_{\lambda=1,2}\left[u^{(\lambda)}(k)b_\lambda(k)e^{-ik\cdot x} + v^{(\lambda)}(k)d_\lambda^\dagger(k)e^{ik\cdot x}\right] \\ \bar{\psi}(x) &= \int \frac{d^3k'}{(2\pi)^3}\frac{m}{\omega_{k'}}\sum_{\lambda'=1,2}\left[\bar{u}^{(\lambda')}(k')b_{\lambda'}^\dagger(k')e^{ik'\cdot x} + \bar{v}^{(\lambda')}(k')d_{\lambda'}^\dagger(k')e^{-ik'\cdot x}\right] \end{aligned}$$

express $Q = \int d^3x j^0(x)$ in terms of creation and annihilation operators in normal order.
(c) Show that the states $b_\lambda^\dagger(k)|0\rangle$ and $d_\lambda^\dagger(k)|0\rangle$ are eigenstates of Q with eigenvalues $+1$ and -1, respectively.

2. In the chiral representation, the matrix $\frac{1}{2}\vec{\Sigma}\cdot\hat{k}$ can be written

$$\frac{1}{2}\vec{\Sigma}\cdot\hat{k} = \frac{1}{2|\vec{k}|}\begin{pmatrix} \vec{\sigma}\cdot\vec{k} & 0 \\ 0 & \vec{\sigma}\cdot\vec{k} \end{pmatrix}. \tag{1}$$

(a) Using the chiral representation, show that

$$\frac{1}{2}\vec{\Sigma}\cdot\hat{k}u(k) = \frac{1}{2}\gamma^5 u(k) \tag{2}$$

in the limit $E \gg m$, where $u(k)$ can be expressed in terms of a particle at rest using

$$u(k) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix} E+m+\vec{\sigma}\cdot\vec{k} & 0 \\ 0 & E+m-\vec{\sigma}\cdot\vec{k} \end{pmatrix} u(0)$$

(b) Show that the projection operators

$$P_\pm = \frac{1\pm\gamma^5}{2}$$

select only the components of $\psi(x)$ that transform under Lorentz transformations as $e^{\pm\vec{\sigma}/2\cdot\vec{\phi}}$.
(c) If $\frac{1}{2}\vec{\Sigma}\cdot\hat{k}$ is interpreted as an operator that gives the projection of the spin along an axis that points in the direction of the momentum vector, show that $P_+\psi(x)$ represents a field with positive helicity and that $P_-\psi(x)$ represents a field with negative helicity.