Physics 565 - Fall 2010, Assignment #1, Due January 22$^{\text{nd}}$

1. Show that the normalization of states:

$$\langle k|k'\rangle = (2\pi)^3 \cdot 2E \cdot \delta^3(\vec{k}-\vec{k}')$$

is invariant under a Lorentz boost in the x-direction.

2. Show that the Lorentz invariant measure

$$\frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2E}$$

can be written in the mainifestly Lorentz covariant form:

$$\frac{d^4k}{(2\pi)^4} \cdot (2\pi)\delta(k^2-m^2)\theta(k^0)$$

where

$$\theta(x) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } x>0 \end{cases}$$

3. Investigate "Klein's Paradox" which implies the unavoidable existance of particle creation and anti-particles when analysing even a simple problem using relativistic quantum mechanics.

Consider a potential of the form:

$$U(x) = \begin{cases} 0 & \text{if } x<0 \\ V & \text{if } x>0 \end{cases}$$

and consider a particle incident from the left which is described by a wavefunction of the form

$$\psi_I(x,t) = e^{-i(Et-kx)}.$$

After scattering from the potential barrier, there will be reflected and transmitted waves

$$\psi_R(x,t) = ae^{-i(Et+kx)}$$
$$\psi_T(x,t) = be^{-i(E't-k'x)}$$

where $E' = E - V$ and a and b are constants to be determined by matching the wavefunction and its derivatives at the boundary.

(a) Derive expressions for a and b and for k' using the classical expression $E = k^2/2m$.

(b) Show that if the classical expression for the energy, $E = k^2/2m$, is used, the transmitted wavefunction dies off exponentially for positive x when $E < V$ and hence, it does not correspond to particle propagation beyond the boundary.

(c) Next, derive expressions for a, b and k' using the relativistic expression $E = \sqrt{k^2+m^2}$ for the energy of a particle with momentum, k.

(d) Under what condition can a particle with energy $E < V$ produce a transmitted wave that propagates freely for $x > 0$, that is, does not die off exponentially as in part (b)?

(e) Discuss how the transmitted wave might be interpreted as an anti-particle state.