The Quark Model

Mesons : $q\bar{q}$ pairs

Baryons : qqq pairs

Quarks have baryon number $B = \frac{1}{3}$

u, d have isospin $\pm\frac{1}{2}$, strangeness 0

s has $I = 0$, strangeness -1.

All have parity -1, spin $\frac{1}{2}$.

This presents some problems. Consider

$$\Delta^{++} = u\uparrow u\uparrow u\uparrow$$

Recall the Fermi exclusion principle ... how can three quarks with identical quantum numbers be in the same state?

Propose a new quantum number, "color".

Quarks carry one of three "color charges".
The strong interaction couples to color charge just like the electromagnetic interaction couples to electric charge.

The observed hadrons are colorless - the colors combine to form "color singlet" states.

What are the baryon wavefunctions?

Four parts: ϕ(space) in position representation
α(spin)
χ(flavor)
η(color)

A totally asymmetric combination of 3 colors is

$$\eta(\text{color}) = \frac{1}{\sqrt{6}}(RGB - RBG + BRG - BGR + GBR - GRB)$$

example, exchange quarks 1 and 2:

$$\eta'(\text{color}) = \frac{1}{\sqrt{6}}(GRB - BRG + RBG - GBR + BGR - RGB)$$

$$= -\eta(\text{color})$$

If the overall wavefunction is antisymmetric then α(spin) χ(flavor) ϕ(space) is symmetric.

Except for orbitally excited baryons ϕ(space) will correspond to the same ground state wavefunction.

So the spin $\frac{1}{2}$ and spin $\frac{3}{2}$ baryons are described in terms of their α(spin) χ(Flavor) wavefunctions.

Easy case: Spin $3/2$. since $\alpha(\text{spin}) = (\uparrow\uparrow\uparrow)$.

Symmetric combinations of 3 quarks:

$$\Delta^{++} = |uuu\rangle$$
$$\Delta^{+} = \frac{1}{\sqrt{3}}\left(|duu\rangle + |udu\rangle + |uud\rangle\right)$$
$$\Delta^{0} = \frac{1}{\sqrt{3}}\left(|ddu\rangle + |dud\rangle + |udd\rangle\right)$$
$$\Delta^{-} = |ddd\rangle$$

Σ^{*+} is the same as Δ^{+} except with d replaced by s:

$$\Sigma^{*+} = \frac{1}{\sqrt{3}}\left(|suu\rangle + |usu\rangle + |uus\rangle\right)$$
$$\Sigma^{*0} = \frac{1}{\sqrt{6}}\left(|sdu\rangle + |sud\rangle + |dsu\rangle + |usd\rangle + |dus\rangle + |uds\rangle\right)$$
$$\Sigma^{*-} = \frac{1}{\sqrt{3}}\left(|sdd\rangle + |dsd\rangle + |dds\rangle\right)$$

Ξ^{*0} is the same as Δ^{0} with $d \to s$:

$$\Xi^{*0} = \frac{1}{\sqrt{3}}\left(|ssu\rangle + |sus\rangle + |uss\rangle\right)$$
$$\Xi^{*-} = \frac{1}{\sqrt{3}}\left(|ssd\rangle + |sds\rangle + |dss\rangle\right)$$

$$\Omega^{-} = |sss\rangle$$

Harder case, spin ½ baryons.
Now α(spin) is not unique. α(spin)χ(flavor) is symmetric:

$$|p\uparrow\rangle = \frac{1}{\sqrt{18}}\Big(2|u\uparrow u\uparrow d\downarrow\rangle + 2|d\downarrow u\uparrow u\uparrow\rangle + 2|u\uparrow d\downarrow u\uparrow\rangle$$
$$- |u\downarrow d\uparrow u\uparrow\rangle - |u\uparrow u\downarrow d\uparrow\rangle - |u\downarrow u\uparrow d\uparrow\rangle$$
$$- |d\uparrow u\downarrow u\uparrow\rangle - |u\uparrow d\uparrow u\downarrow\rangle - |d\uparrow u\uparrow u\downarrow\rangle\Big)$$

Wavefunctions for n, Σ^+, Σ^-, Ξ^0, Ξ^- are obtained from symmetry arguments,

eg, $|n\uparrow\rangle$ obtained with $u \Leftrightarrow d$
$|\Xi^0\uparrow\rangle$ obtained with $u \leftrightarrow s$.

$$|\Sigma^0\rangle = \frac{1}{6}\Big(2|s\downarrow d\uparrow u\uparrow\rangle - |s\uparrow d\downarrow u\uparrow\rangle - |s\uparrow d\uparrow u\downarrow\rangle + \text{permutations}\Big)$$

$$|\Lambda^0\rangle = \frac{1}{\sqrt{12}}\Big(|u\uparrow d\downarrow s\uparrow\rangle - |u\downarrow d\uparrow s\uparrow\rangle + \text{permutations}\Big)$$

With these wavefunctions we can predict

- masses
- magnetic moments
- electromagnetic transitions.

Simpler case is the meson wavefunctions: spin 0, odd parity

$$\pi^+ = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)(u\bar{d}) = \frac{1}{\sqrt{2}}\left(|u\uparrow\bar{d}\downarrow\rangle - |u\downarrow\bar{d}\uparrow\rangle\right)$$

$$\pi^0 = \frac{1}{2}(\uparrow\downarrow - \downarrow\uparrow)(u\bar{u} - d\bar{d})$$

$$= \frac{1}{2}\left(|u\uparrow\bar{u}\downarrow\rangle - |d\uparrow\bar{d}\downarrow\rangle - |u\downarrow\bar{u}\uparrow\rangle + |d\downarrow\bar{d}\uparrow\rangle\right)$$

$$\pi^- = \frac{1}{\sqrt{2}}\left(|d\uparrow\bar{u}\downarrow\rangle - |d\downarrow\bar{u}\uparrow\rangle\right)$$

$$K^+ = \frac{1}{\sqrt{2}}\left(|u\uparrow\bar{s}\downarrow\rangle - |u\downarrow\bar{s}\uparrow\rangle\right)$$

$$K^0 = \frac{1}{\sqrt{2}}\left(|d\uparrow\bar{s}\downarrow\rangle - |d\downarrow\bar{s}\uparrow\rangle\right)$$

$$\bar{K}^0 = \frac{1}{\sqrt{2}}\left(|s\uparrow\bar{d}\downarrow\rangle - |s\downarrow\bar{d}\uparrow\rangle\right)$$

$$K^- = \frac{1}{\sqrt{2}}\left(|s\uparrow\bar{u}\downarrow\rangle - |s\downarrow\bar{u}\uparrow\rangle\right)$$

The singlet states are interesting:

η and η' are isospin 0, spin 0, negative parity. Two ways to write such a wavefunction:

$$\psi_1 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

$$\psi_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

Physical states are mixtures of these:

$$\eta = \psi_8 \sin\theta + \psi_1 \cos\theta$$
$$\eta' = \psi_8 \cos\theta - \psi_1 \sin\theta$$

The vector mesons are the easiest:

$$\rho^+ = |u\uparrow \bar{d}\uparrow\rangle$$

$$\rho^0 = \frac{1}{\sqrt{2}}\left(|u\uparrow \bar{u}\uparrow\rangle - |d\uparrow \bar{d}\uparrow\rangle\right)$$

$$\rho^- = -|d\uparrow \bar{u}\uparrow\rangle$$

$$K^{*+} = |u\uparrow \bar{s}\uparrow\rangle$$

$$K^{*0} = |d\uparrow \bar{s}\uparrow\rangle$$

$$\overline{K^{*0}} = |s\uparrow \bar{d}\uparrow\rangle$$

$$K^{*-} = |s\uparrow \bar{u}\uparrow\rangle$$

Now the ω and the ϕ are interesting:

$$\omega = \psi_8 \sin\theta_v + \psi_1 \cos\theta_v$$

$$\phi = -\psi_8 \cos\theta_v + \psi_1 \sin\theta_v$$

It turns out that $\theta_v \approx 35^\circ \Rightarrow \sin\theta_v = 0.57$
$\cos\theta_v = 0.82$

$$\phi = -0.82\left(\frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{2.45}\right) + 0.57\left(\frac{u\bar{u} + d\bar{d} + s\bar{s}}{1.73}\right)$$

$$= -0.335\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right) + 0.329\left(u\bar{u} + d\bar{d} + s\bar{s}\right)$$

$$= -0.005\left(u\bar{u} + d\bar{d}\right) + 0.999\left(s\bar{s}\right)$$

ie, the ϕ is almost completely an $s\bar{s}$ state

$$\omega \approx \frac{1}{\sqrt{2}}\left(u\bar{u} + d\bar{d}\right)$$

Meson masses.

The potential energy associated with two magnetic dipoles is called the HyperFine splitting

$$\Delta E_{Hyp} = -\frac{2}{3}\vec{\mu}_1\cdot\vec{\mu}_2\,|\psi(0)|^2$$

$$= \frac{8\pi\alpha}{3}\,\frac{\vec{S}_1\cdot\vec{S}_2}{m_1 m_2}\,|\psi(0)|^2$$

We propose to model hadron masses by

$$m_{q_1\bar{q}_2} = m_1 + m_2 + a\,\frac{\vec{S}_1\cdot\vec{S}_2}{m_1 m_2}$$

$$m_{q_1 q_2 q_3} = m_1 + m_2 + m_3 + \frac{a'}{2}\left[\frac{\vec{S}_1\cdot\vec{S}_2}{m_1 m_2} + \frac{\vec{S}_1\cdot\vec{S}_3}{m_1 m_3} + \frac{\vec{S}_2\cdot\vec{S}_3}{m_2 m_3}\right]$$

Recall that $S^2 = (\vec{S}_1+\vec{S}_2)^2 = S_1^2 + S_2^2 + 2\vec{S}_1\cdot\vec{S}_2$

So $\vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}S(S+1) - \frac{1}{2}\cdot\frac{3}{2} - \frac{1}{2}\cdot\frac{3}{2}$

$$= \begin{cases} +1/4 & \text{when } S = 1 \\ -3/4 & \text{when } S = 0 \end{cases}$$

Try it out with the mesons.

$$\left.\begin{aligned} m_{K^+} &= m_u + m_s - \frac{3a}{4m_u m_s} \\ m_{K^{*+}} &= m_u + m_s + \frac{a}{4m_u m_s} \\ m_{\pi^+} &= m_u + m_d - \frac{3a}{4m_u m_d} \\ m_{\rho^+} &= m_u + m_d + \frac{a}{4m_u m_d} \end{aligned}\right\}$$

$$\frac{m_{K^*} - m_K}{m_\rho - m_\pi} = \frac{m_d}{m_s} \approx 0.63$$

$$\frac{892 - 495}{770 - 139} = 0.63.$$

To apply this to the baryons we need to consider how the spins and flavor wavefunctions were constructed.

eg. Σ are isospin 1

Λ is isospin 0.

Hence, $\Sigma = \frac{1}{\sqrt{2}}(uu + dd)s$

$\Lambda = \frac{1}{\sqrt{2}}(uu - dd)s$

The spin wavefunctions are

$$\Sigma = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)\uparrow \qquad \Sigma^* = \uparrow\uparrow\uparrow$$

$$\Lambda = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow$$

so that the product of spin and flavor wavefunctions will be symmetric.

Consider $(\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2 = (\vec{S}_1 + \vec{S}_2)^2 + 2(\vec{S}_1 + \vec{S}_2)\cdot\vec{S}_3 + \vec{S}_3^{\,2}$

$$(\vec{S}_1 + \vec{S}_2)\cdot\vec{S}_3 = \frac{1}{2}\left[(\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2 - (\vec{S}_1 + \vec{S}_2)^2 - S_3^2\right]$$

For the Λ,
$$(\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2 = s(s+1) = \frac{1}{2}\cdot\frac{3}{2}$$
$$(\vec{S}_1 + \vec{S}_2)^2 = S_{12}(S_{12}+1) = 0$$
$$S_3^2 = \frac{1}{2}\cdot\frac{3}{2}$$

So $(\vec{S}_1 + \vec{S}_2)\cdot\vec{S}_3 = \begin{cases} 0 & \text{for } \Lambda \\ -1 & \text{for } \Sigma \\ +\frac{1}{2} & \text{for } \Sigma^* \end{cases}$

So $m_{\Sigma,\Sigma^*,\Lambda} = 2m_u + m_s + a'\dfrac{\vec{S}_1\cdot\vec{S}_2}{m_u^2} + a'\dfrac{(\vec{S}_1 + \vec{S}_2)\cdot\vec{S}_3}{m_u m_s}$

$$m_\Lambda = 2m_u + m_s - \frac{3a'}{4m_u^2}$$

$$m_\Sigma = 2m_u + m_s + \frac{a'}{4m_u^2} - \frac{a'}{m_u m_s}$$

$$m_{\Sigma^*} = 2m_u + m_s + \frac{a'}{4m_u^2} + \frac{a'}{2m_u m_s}$$

Solve for m_u, m_s, a' given the masses of the baryons.

$$\Rightarrow \quad m_{u,d} = 363 \text{ MeV}$$

$$m_s = 538 \text{ MeV}$$

$$\frac{m_d}{m_s} = 0.67$$

Magnetic Moments

$$\mu_i = Q_i \left(\frac{e}{2m_i} \right)$$

$$\vec{\mu}_z = Q_i \left(\frac{e}{m_i} \right) S_z = Q_i \left(\frac{e}{2m_i} \right) (\sigma_z)_i$$

where $(\sigma_z)_i = \pm 1$ if quark i has spin $\uparrow$ or $\downarrow$.

Example:

$$\mu_p = \sum_{i=1}^{3} \langle p \uparrow | \mu_z | p \uparrow \rangle$$

$$= \frac{1}{18} \left(4(2\mu_u - \mu_d) + \mu_d + \mu_d \right) \times 3$$

$$= \frac{1}{3} \left(4\mu_u - 2\mu_d + \mu_d \right)$$

$$= \frac{1}{3} (4\mu_u - \mu_d)$$

Similarly, $\mu_n = \frac{1}{3}(4\mu_d - \mu_u)$

Since $m_u \approx m_d$, $\quad \dfrac{\mu_u}{\mu_d} \approx \dfrac{Q_u}{Q_d} = \dfrac{+2/3}{-1/3} = -2.$

So $\quad \dfrac{\mu_n}{\mu_p} = \dfrac{4\mu_d - \mu_u}{4\mu_u - \mu_d} = \dfrac{4 - \mu_u/\mu_d}{4\mu_u/\mu_d - 1} = \dfrac{6}{-9} = -\dfrac{2}{3}$

Compare with experiment:

$$\frac{\mu_n}{\mu_p} = -0.685$$

Not bad.