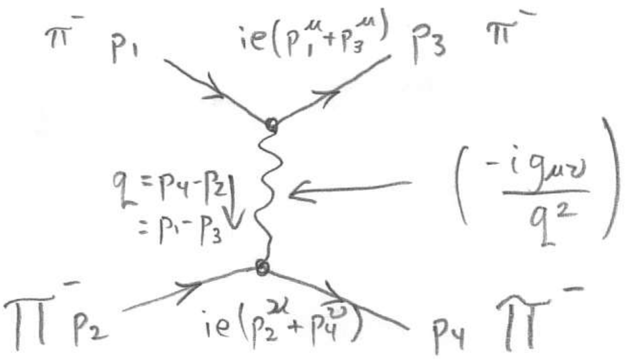
Oct 18, 2005

Feynman Rules for spinless particles.

We derived the rules that allowed us to calculate processes like

$$\pi^- + \Pi^- \rightarrow \pi^- + \Pi^-$$

where π and Π are distinct charged spinless particles with masses m and M.



$$d\sigma = \frac{|\mathcal{M}|^2}{F} dQ$$

where $F = 4\left((p_1 \cdot p_2)^2 - m_1^2 m_2^2\right)^{1/2}$

$$dQ = (2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2) \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4}$$

We worked out that

$$-i\mathcal{M} = \left(ie(p_3^\mu + p_1^\mu)\right)\left(-\frac{i g_{\mu\nu}}{q^2}\right)\left(ie(p_4^\nu + p_2^\nu)\right)$$

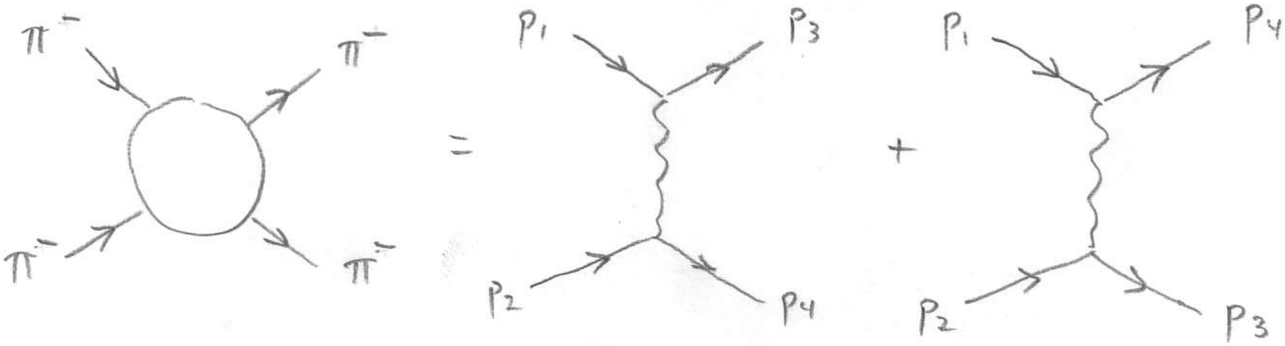
$$= ie^2\left(\frac{s-u}{t}\right)$$

When particle 1 has momentum $p_1 = (E_1, \vec{p}_1)$ and particle 2 is at rest in the lab, $p_2 = (M, \vec{0})$

then $F = 4M|\vec{p}_1|$

$$dQ = \frac{|\vec{p}_3|\, d\Omega}{4E_4 (2\pi)^2} \Rightarrow \frac{d\sigma}{d\Omega} = \begin{cases} \dfrac{\alpha^2}{16 K^2 \sin^4 \theta/2} & p \ll m \\ \dfrac{\alpha^2}{4E^2 \sin^4 \theta/2} & p \gg m \end{cases}$$

What if the two particles are identical?



We add together all possible invariant amplitudes that give the same final state.

$$-i\mathcal{M}_1 = ie^2 \left(\frac{s-u}{t} \right)$$

Diagram 2 is the same as 1 except with $p_3 \leftrightarrow p_4$. So

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \rightarrow s$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \rightarrow (p_1 - p_4)^2 = u$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \rightarrow (p_1 - p_3)^2 = t$$

So $-i\mathcal{M}_2 = ie^2 \left(\dfrac{s-t}{u} \right)$

$$-i\mathcal{M} = -ie^2\left(\frac{u-s}{t} + \frac{t-s}{u}\right)$$ scattering identical charged particles.

What about antiparticles?

Recall that the charge current density was

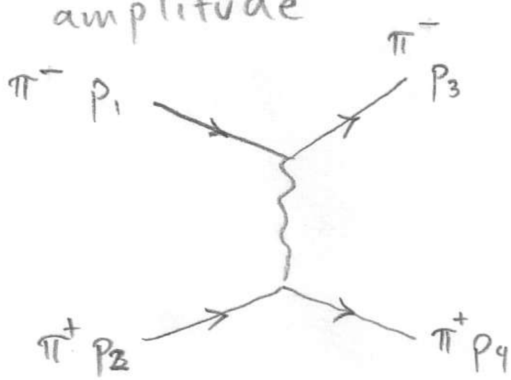
$$j^\mu = -2ep^\mu$$

A negative energy solution described an antiparticle with charge current density

$$j^\mu = -2e(-p^\mu)$$

where p^μ is the physical 4-momentum.

So $\pi^+\pi^- \to \pi^+\pi^-$ is described by the invariant amplitude



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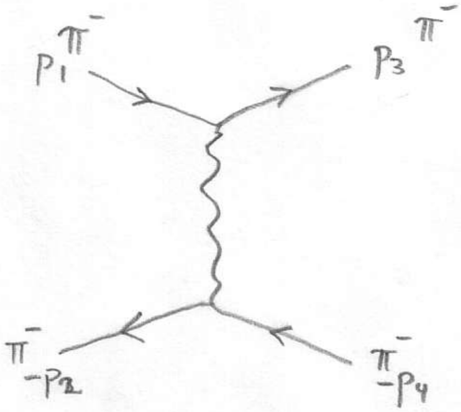
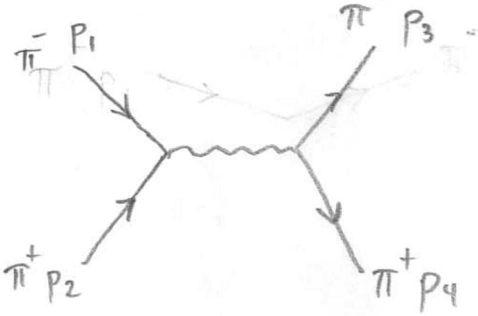


$p_2 \to -p_2$
$p_4 \to -p_4$



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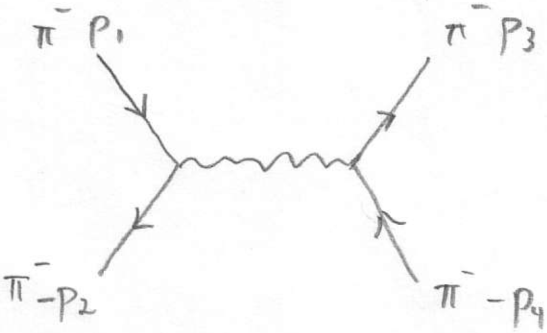

$p_3 \to -p_2$ $\quad p_2 \to -p_4$
$p_4 \to p_3$

The original $\pi^-\pi^- \to \pi^-\pi^-$ amplitude was

$$-iM = -ie^2\left(\frac{u-s}{t} + \frac{t-s}{u}\right)$$

For $\pi^-\pi^+ \to \pi^-\pi^+$ it turns out to be

$$-iM = -ie^2\left(-\left(\frac{u-s}{t}\right) + \left(\frac{t-u}{s}\right)\right)$$

$$= -ie^2\left(\frac{s-u}{t} + \frac{t-u}{s}\right)$$

Consider $\pi^+\pi^- \to \pi^+\pi^-$ in the center of mass frame.

$$p_1 = (E_1, \vec{p}_i) \qquad p_3 = (E_3, \vec{p}_3)$$
$$p_2 = (E_2, -\vec{p}_i) \qquad p_4 = (E_4, \vec{p}_4)$$

Flux factor, $F = 4(|\vec{p}_i|E_2 + |\vec{p}_i|E_1)$

$$= 4|\vec{p}_i|(E_1 + E_2)$$
$$= 4|\vec{p}_i|\sqrt{s} \qquad (*)$$

Phase space, $dQ = (2\pi)^4\delta^4(p_3 + p_4 - p_1 - p_2)\dfrac{d^3p_3}{(2\pi)^3 \cdot 2E_3}\dfrac{d^3p_4}{(2\pi)^3 \cdot 2E_4}$

$$= (2\pi)^{-2}\delta(E_3 + E_4 - E_1 - E_2) \cdot \frac{d^3p_3}{4E_3E_4}$$

$$= (2\pi)^{-2}\delta(E_3 + E_4 - E_1 - E_2)\frac{|\vec{p}_3|^2\,dp_3\,d\Omega}{4E_3E_4}$$

But since $E_3^2 = |\vec{p}_3|^2 + m_3^2$

$$E_3\,dE_3 = |\vec{p}_3|\,dp_3$$

So $dQ = \dfrac{1}{(4\pi)^2}\delta(E_3 + E_4 - E_1 - E_2)\dfrac{|\vec{p}_3|\,dE_3\,d\Omega}{E_4}$

In the c.m. frame,

$$E_3^2 = |\vec{p}_f|^2 + m_3^2 \qquad E_4^2 = |\vec{p}_f|^2 + m_4^2$$

$$E_3 \, dE_3 = E_4 \, dE_4 = |\vec{p}_f| \, dp_f$$

If we write $W = E_3 + E_4$

then $dW = dE_3 + dE_4$

$$= \frac{|\vec{p}_f| \, dp_f}{E_3} + \frac{|\vec{p}_f| \, dp_f}{E_4}$$

$$= \frac{W \, |\vec{p}_f| \, dp_f}{E_3 E_4}$$

But $\dfrac{|\vec{p}_f| \, dp_f}{E_3} = dE_3$

So $dW = \dfrac{W \, dE_3}{E_4}$

So $$dQ = \frac{1}{(4\pi)^2} \delta(E_3 + E_4 - E_1 - E_2) \frac{|\vec{p}_f|}{E_4} dE_3 \, d\Omega$$

$$= \frac{1}{(4\pi)^2} \delta(W - \sqrt{s}) |\vec{p}_f| \frac{dW}{W} d\Omega$$

$$= \frac{1}{(4\pi)^2} \frac{|\vec{p}_f|}{\sqrt{s}} d\Omega \qquad (*)$$

To summarize, in the cm frame for $2 \to 2$ scattering,

$$F = 4 |\vec{p}_i| \sqrt{s}$$

$$dQ = \frac{1}{(4\pi)^2} \frac{|\vec{p}_f|}{\sqrt{s}} d\Omega$$

So $$d\sigma = \frac{|\mathcal{M}|^2}{64\pi^2 s} \frac{|\vec{p}_f|}{|\vec{p}_i|} d\Omega$$

If we write $\vec{p}_i \cdot \vec{p}_f = |\vec{p}_i||\vec{p}_f| \cos\theta$ then

$$s = (E_1 + E_2)^2 = 4E^2 \quad \text{(since } \pi^+ \text{ and } \pi^- \text{ have equal mass)}$$

$$\begin{aligned} t = (p_3 - p_1)^2 &= p_1^2 + p_3^2 - 2p_1 \cdot p_3 \\ &= 2m^2 - 2E^2 + 2p^2 \cos\theta \\ &= -2p^2(1 - \cos\theta) \end{aligned}$$

$$u = (p_4 - p_1)^2 = -2p^2(1 + \cos\theta)$$

Also, $|\vec{p}_i| = |\vec{p}_f| = p$.

$$\mathcal{M} = e^2 \left(\frac{s-u}{t} + \frac{t-u}{s} \right)$$

$$= e^2 \left(\frac{4E^2 + 2p^2(1+\cos\theta)}{-2p^2(1-\cos\theta)} + \frac{4p^2\cos\theta}{4E^2} \right)$$

At high energies ($E^2 \approx p^2$) this becomes

$$\mathcal{M} = e^2\left(-\left(\frac{3 + \cos\theta}{1-\cos\theta}\right) + \cos\theta\right)$$

$$= e^2\left(\frac{\cos\theta - \cos^2\theta - 3 - \cos\theta}{1-\cos\theta}\right)$$

$$= -e^2\left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)$$

$$|\mathcal{M}|^2 = e^4\left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)^2$$

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}\left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)^2$$

$$= \frac{\alpha^2}{4s}\left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)^2$$

Notice that $\frac{d\sigma}{d\Omega}$ diverges when $\theta \to 0$.

this corresponds to the case where no interaction takes place, ie, no momentum is exchanged between the particles.

Also notice that the cross section decreases like $1/E_{cm}^2$.