Hadron Spectroscopy

The $J/\psi$ and its excited states were identified as bound states of $c\bar{c}$ quarks. In 1977, 78 a similar set of resonances was discovered in proton fixed target experiments at Fermilab (400 GeV protons on a Cu/Pb target). The $\Upsilon$ resonances were associated with $b\bar{b}$ production.

\[
\begin{array}{ccc}
\text{c}\bar{c} & ^{2S+1}L_J & \text{mass} \\
\eta_c & ^1S_0 & 2980 \text{ MeV} \\
J/\psi & ^3S_1 & 3097 \\
\chi_{c_0}(1P) & ^3P_0 & 3415 \\
\chi_{c_1}(1P) & ^3P_1 & 3511 \\
\chi_{c_2}(1P) & ^3P_2 & 3556 \\
\Psi(2S) & ^3S_1 & 3686 \\
b\bar{b} & & \\
\eta_b & ^1S_0 & ? \\
\Upsilon(1S) & ^3S_1 & 9460 \text{ MeV} \\
\chi_{b_0}(1P) & ^3P_0 & 9860 \text{ MeV} \\
\chi_{b_1}(1P) & ^3P_1 & 9893 \text{ MeV} \\
\chi_{b_2}(1P) & ^3P_2 & 9913 \\
\chi_{b_0}(2P) & ^3P_0 & 10.023 \text{ GeV} \\
\chi_{b_1}(2P) & ^3P_1 & 10.232 \\
\chi_{b_2}(2P) & ^3P_2 & \\
\Upsilon(3S) & & \\
\Upsilon(4S) & & 10.580 \text{ GeV} \\
\Upsilon(6S) & & \\
\end{array}
\]
The rich structure in the mass spectrum is due to spin-spin and spin-orbit interactions. 

The spin-spin splitting is of the form

\[ \Delta H_{\text{hyp}} = \frac{\alpha}{m_1 m_2} \left[ \frac{\hbar^2}{2} \frac{8 \pi \hbar}{3} \vec{s}_1 \cdot \vec{s}_2 + \frac{1}{r^3} S_{12} \right] \]

The \( \vec{s}_1 \cdot \vec{s}_2 \) term only acts on the S-wave states since these are the only ones for which the radial wavefunctions are finite at \( r = 0 \). It has expectation values:

\[ \langle \,^3S_0 \mid \vec{s}_1 \cdot \vec{s}_2 \mid \,^3S_0 \rangle = -\frac{3}{4} \]
\[ \langle \,^3S_1 \mid \vec{s}_1 \cdot \vec{s}_2 \mid \,^3S_1 \rangle = +\frac{1}{4} \]

(Recall \( \vec{s}_1 \cdot \vec{s}_2 = \frac{1}{2} \left[ S(S+1) - S_1(S+1) - S_2(S+1) \right] \))

The \( S_{12} \) is called the "Tensor operator" and it has expectation values:

\[ \langle \,^3P_2 \mid S_{12} \mid \,^3P_2 \rangle = -\frac{2}{5} \]
\[ \langle \,^3P_1 \mid S_{12} \mid \,^3P_1 \rangle = 2 \]
\[ \langle \,^3P_0 \mid S_{12} \mid \,^3P_0 \rangle = -4 \]
\[ \langle \,^1S_0 \mid S_{12} \mid \,^1S_0 \rangle = 0 \]
\[ \langle \,^3S_0 \mid S_{12} \mid \,^3S_0 \rangle = 0 \]
The spin-orbit interaction has the form:

$$\Delta H_{\text{s.o.}} = -\frac{1}{2r} \frac{\partial V}{\partial \mathbf{r}} \left[ \frac{\mathbf{L} \cdot \mathbf{S}_1}{m_1^2} + \frac{\mathbf{L} \cdot \mathbf{S}_2}{m_2^2} \right]$$

$$+ \frac{\alpha}{r^3} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \left( \frac{\mathbf{L} \cdot \mathbf{S}_1}{m_1^2} + \frac{\mathbf{L} \cdot \mathbf{S}_2}{m_2^2} \right)$$

Recall $$\mathbf{L} \cdot \mathbf{S} = \frac{1}{2} \left[ J(J+1) - L(L+1) - S(S+1) \right]$$

for quarks with equal mass,

$$\frac{\mathbf{L} \cdot \mathbf{S}_1}{m_1^2} + \frac{\mathbf{L} \cdot \mathbf{S}_2}{m_2^2} = \frac{1}{m_a^2} \mathbf{L} \cdot \mathbf{S}$$

where $$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$$.

$$\langle 3p_2 | \mathbf{L} \cdot \mathbf{S} | 3p_2 \rangle = +1$$

$$\langle 3p_1 | \mathbf{L} \cdot \mathbf{S} | 3p_1 \rangle = -1$$

$$\langle 3p_0 | \mathbf{L} \cdot \mathbf{S} | 3p_0 \rangle = -2$$

$$H_{\text{s.o.}}$$ also depends on the shape of the potential that binds the quarks together.
If heavy vector mesons (c\bar{c} or b\bar{b}) can really be thought of as nonrelativistic systems like the Hydrogen atom, then what is the form of the potential that binds the quarks together?

For Hydrogen it is \( V(r) \sim -\frac{1}{r} \).

Is it the same for Q\bar{Q} states?

It was an empirical observation that the partial widths of the \( ^3S_1 \) states decaying to \( e^+e^- \) were the same for all vector mesons, after adjusting for the quark charges:

\[
\frac{\Gamma \left( ^3S_1 \rightarrow e^+e^- \right)}{Q_a^2} \approx 10 \text{ keV}
\]

for the \( \rho, \omega, \phi, J/\psi \) and \( \Upsilon \) states.

This process can be calculated:

\[
\Gamma \left( ^3S_1 \rightarrow e^+e^- \right) = 16 \pi \alpha^2 \frac{Q_a^2}{m_v^2} \left| \frac{\psi(0)}{m_v} \right|^2
\]

where \( m_v \) is the mass of the vector meson. This is called the Van Royen-Weisskopf formula.
The fact that \( \frac{\Gamma}{Q_a^2} \approx 10 \text{ keV} \)
tells us that \( \frac{|\Psi(0)|^2}{m_v^2} \)
the same for all vector mesons.
So \( |\Psi(0)|^2 \propto m_v^2 \).
Recall that for a Hydrogen atom
\[ \Psi(r) = \frac{1}{\sqrt{\pi}} \left( \frac{1}{a_o} \right)^{3/2} e^{-r/a_o} \]
where \( a_o = \frac{1}{\alpha \text{me}} \) or \( \frac{1}{\alpha m_e/2} \)
 reduced mass
So for the Hydrogen atom \( |\Psi(0)|^2 \propto m^3 \).
For the harmonic oscillator problem
the spatial representation of the ground state wavefunction was
\[ \Psi(x) = \left( \frac{\beta^2}{\pi} \right)^{1/4} e^{-\beta^2 x^2/2} \]
where \( \beta = \sqrt{\kappa m} \).
So in this case \( |\Psi(0)|^2 \propto m^{3/2} \).
The fact that $|\psi(0)|^2 \propto m^2$ for vector mesons suggests that the form of the potential is somewhere in between the coulomb potential and the harmonic oscillator potential.

\[
\text{Coulomb} \quad \begin{array}{c}
\text{Q}\bar{Q} \\
\text{Harmonic Oscillator}
\end{array}
\]

Phenomenologists usually parameterize the potential as being linear + coulomb:

\[
V(r) \sim -\frac{4/3\alpha_s}{r} + c + br
\]

where $4/3\alpha_s$ replaces the electromagnetic coupling constant $\alpha$ in the coulomb potential.

We know that $\alpha = \frac{1}{137}$ for low energy EM processes. What is $\alpha_s$?
Someone calculated the partial width for the process $V \to \gamma \gamma \gamma$:

$$
\Gamma(3S_1 \to \gamma \gamma \gamma) = \frac{2(\pi^2 - 9)}{9\pi} \alpha_s^6 m
$$

The equivalent of the photon for strong interactions is the gluon:

$$
\Gamma(3S_1 \to \text{hadrons}) = \frac{2(\pi^2 - 9)}{9\pi} \left(\frac{4}{3}\right)^6 \alpha_s^6 m
$$

Hence, $\alpha_s^6 = \frac{\Gamma(3S_1 \to \text{hadrons})}{M_c} \cdot \frac{9\pi}{2(\pi^2 - 9)} \left(\frac{3}{4}\right)^6$

The total width of the $J/\psi$ is 87 keV and the hadronic branching fraction is 88%. This gives

$$
\alpha_s^6 = \frac{0.222}{M_c}
$$

If $M_c \approx \frac{1}{2} M_{J/\psi}$ this gives

$$
\alpha_s = \left(\frac{0.087 \text{ MeV} \times 0.88}{1500 \text{ MeV}} \times \frac{9\pi}{2(\pi^2 - 9)}\right)^{\frac{1}{6}} \times \frac{3}{4}
$$

$$
= 0.23
$$

So $\alpha_s \gg \alpha$. Maybe that's why strong interactions are so strong.