1. For a tensor with two indices, contraction is the same as matrix multiplication.

(a) When $L^\mu{}_\nu$ is represented as a matrix with μ labeling the row and ν labeling the column,

$$L^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

we can write

$$g_{\rho\sigma} L^\sigma{}_\nu = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ -\gamma\beta & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and $$L^\rho{}_\mu g_{\rho\sigma} L^\sigma{}_\nu = \begin{pmatrix} \gamma^2(1-\beta^2) & 0 & 0 & 0 \\ 0 & -\gamma^2(1-\beta^2) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

But $\gamma^2(1-\beta^2) = 1$ so

$$L^\rho{}_\mu g_{\rho\sigma} L^\sigma{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}$$

(b) If $a'^{\mu} = L^{\mu}{}_{\rho} a^{\rho}$ and $b'^{\nu} = L^{\nu}{}_{\sigma} b^{\sigma}$ then

$$
\begin{aligned}
a' \cdot b' &= g_{\mu\nu} a'^{\mu} b'^{\nu} \\
&= g_{\mu\nu} L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} a^{\rho} b^{\sigma} \\
&= g_{\rho\sigma} a^{\rho} b^{\sigma} \\
&= a \cdot b
\end{aligned}
$$

(c) We can write $a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$ and since a^0, b^0 and $\vec{a} \cdot \vec{b}$ are invariant under rotations in 3-dimensional space we can always choose a coordinate transformation such that any boost is along the x-axis.

In such a case the result from (b) can be applied to conclude that $a' \cdot b' = a \cdot b$ in all inertial reference frames.

2. (a) The velocity of particle A can be chosen to be in the +z direction. Furthermore, the momentum of particle a can be chosen to lie in the x-z plane.

Then, in the rest frame of A:

$$p_a^* = (E_a^*, |\vec{p}_a^*| \sin\theta^*, 0, |\vec{p}_a^*| \cos\theta^*)$$

In the lab frame,

$$p_{ax} = p_{ax}^* = |\vec{p}_a^*| \sin\theta^*$$

$$p_{az} = \gamma |\vec{p}_a^*| \cos\theta^* + \gamma\beta E_a^*$$

then, $\tan\theta = \dfrac{p_{ax}}{p_{az}} = \dfrac{\sin\theta^*}{\gamma\cos\theta^* + \gamma\beta/\beta^*}$

where $\beta^* = |\vec{p}_a^*| / E_a^*$.

If $\beta/\beta^* < 1$ then $\theta^* \to \pi$ will maximize θ with a value of π.

However, if $\beta/\beta^* > 1$ then θ will be maximal when $\dfrac{d\theta}{d\theta^*} = 0$.

$$\theta = \tan^{-1}\left(\frac{\sin\theta^*}{\gamma\cos\theta^* + \gamma\beta/\beta^*}\right) \equiv \tan^{-1} x$$

Then, $$\frac{d\theta}{d\theta^*} = \left(\frac{1}{1+x^2}\right)\left(\frac{\cos\theta^*}{\gamma\cos\theta^* + \gamma\beta/\beta^*} + \frac{\gamma\sin^2\theta^*}{(\gamma\cos\theta^* + \gamma\beta/\beta^*)^2}\right)$$

$$= \left(\frac{1}{1+x^2}\right)\left(\frac{\gamma + \gamma\beta/\beta^* \cos\theta^*}{(\gamma\cos\theta^* + \gamma\beta/\beta^*)^2}\right) = 0.$$

Hence, $\frac{d\theta}{d\theta^*} = 0$ when $1 + \beta/\beta^* \cos\theta^* = 0$

That is, $\cos\theta^* = -\frac{\beta^*}{\beta}$.

(b) Since $p_z = \gamma|\vec{p}_a^{\,*}|\cos\theta^* + \gamma\beta E_a^*$, the minimum p_z will be $p_z^{min} = \gamma\beta E_a^* - \gamma|\vec{p}_a^{\,*}|$.

If $p_z^{min} = 0$ then

$$\gamma\beta E_a^* - \gamma|\vec{p}_a^{\,*}| = 0$$

or $\beta - \beta^* = 0$

where $\beta^* = |\vec{p}_a^{\,*}|/E_a^*$.

Therefore, the minimum velocity of particle A for which particle a always moves with $p_z > 0$ will be

$$\beta > \beta^* = |\vec{p}_a^{\,*}|/E_a^*.$$