Physics 56400 Assignment #3 – Due October 24th

1. Suppose a beam of protons has a Gaussian distribution of intensity of the form

$$I(r) = I_0 \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2}$$

   where $I_0 = 10^6 \text{ s}^{-1}$ and $\sigma = 1 \text{ mm}$. Suppose this beam were incident at right-angles to a target foil made of a $100\ \mu\text{m}$ thick sheet of pure ^{7}Li. If the cross section for $p + {}^7Li \rightarrow {}^7 Be + n$ is 10 mb, calculate the rate at which neutrons are produced.

2. Calculate the relative cross sections for inclusive Σ^+ and Σ^- production from
   (a) A beam of K^- incident on a hydrogen target,
   (b) A beam of K^- incident on a deuterium target.

3. Using the empirical mass formula for the baryons:

$$M_{q_1q_2q_3} = m_{q_1} + m_{q_2} + m_{q_3} + \kappa\frac{\vec{S}_1 \cdot \vec{S}_2}{m_{q_1}m_{q_2}} + \kappa\frac{\vec{S}_2 \cdot \vec{S}_3}{m_{q_2}m_{q_3}} + \kappa\frac{\vec{S}_1 \cdot \vec{S}_3}{m_{q_1}m_{q_3}}$$

   with parameters

$$m_u = m_d = 364\ MeV,$$
$$m_s = 535\ MeV$$
$$\kappa = 0.0259 \times 10^9\ MeV^3$$

   Calculate the masses of the spin 1/2 and spin 3/2 baryons and compare your results with their measured values. Show, as an example, the calculation for the Σ^{*+} baryon, but use a spreadsheet to perform similar calculations for the other baryons.

Assignment # 3

1. The rate per unit area of the incident beam is

$$\frac{dR}{dA} = \frac{N_A \rho}{m} I(r)\, \sigma_0 \,\Delta z$$

where Δz is the thickness of the target. Thus, the total rate is

$$R = \frac{N_A \rho}{m} \sigma_0 \,\Delta z \int I(r)\, dA$$

$$= \frac{N_A \rho}{m} \sigma_0 \,\Delta z \cdot \frac{I_0}{2\pi\sigma^2} \int e^{-(x^2+y^2)/2\sigma^2}\, dx\, dy$$

$$= \frac{N_A \rho}{m} \sigma_0 I_0 \,\Delta z$$

$$= \frac{(6.02\times10^{23}\ \mathrm{mol}^{-1})(0.534\ \mathrm{g/cm^3})(100\times10^{-4}\ \mathrm{cm})(10^6\ \mathrm{s}^{-1})(10\times10^{-27}\ \mathrm{cm}^2)}{(7.016\ \mathrm{g/mol})}$$

$$= 4.58\times10^{-6}\ \mathrm{s}^{-1}$$

2. The $K^- p$ system has

$$\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{2}}|1\,0\rangle + \tfrac{1}{\sqrt{2}}|0\,0\rangle$$

Final states can be

$$|\Sigma^+\pi^+\rangle = |1\,1\rangle|1\,1\rangle = |2\,2\rangle$$

$$|\Sigma^+\pi^0\rangle = |1\,1\rangle|1\,0\rangle = \tfrac{1}{\sqrt{2}}|2\,1\rangle + \tfrac{1}{\sqrt{2}}|1\,1\rangle$$

$$|\Sigma^+\pi^-\rangle = |1\,1\rangle|1\,-1\rangle = \tfrac{1}{\sqrt{6}}|2\,0\rangle + \tfrac{1}{\sqrt{2}}|1\,0\rangle + \tfrac{1}{\sqrt{3}}|0\,0\rangle$$

Thus, $\langle\Sigma^+\pi^+|K^-p\rangle = 0$

$$\langle\Sigma^+\pi^0|K^-p\rangle = 0$$

At a given energy, one or the other states with isospin 1 or 0 will dominate.
Thus,

$$\langle\Sigma^+\pi^-|K^-p\rangle_{I=1} = \tfrac{1}{2}$$

$$\langle\Sigma^+\pi^-|K^-p\rangle_{I=0} = \frac{1}{\sqrt{6}}.$$

When there is a Σ^- in the final state

$$|\Sigma^-\pi^+\rangle = |1\,-1\rangle|1\,1\rangle = \tfrac{1}{\sqrt{6}}|2\,0\rangle - \tfrac{1}{\sqrt{2}}|1\,0\rangle + \tfrac{1}{\sqrt{3}}|0\,0\rangle$$

$$|\Sigma^-\pi^0\rangle = |1\,-1\rangle|1\,0\rangle = \tfrac{1}{\sqrt{2}}|2\,-1\rangle - \tfrac{1}{\sqrt{2}}|1\,-1\rangle$$

$$|\Sigma^-\pi^-\rangle = |1\,-1\rangle|1\,-1\rangle = |2\,-2\rangle$$

Thus, $\langle\Sigma^-\pi^+|K^-p\rangle_{I=1} = -\tfrac{1}{2}$

$$\langle\Sigma^-\pi^+|K^-p\rangle_{I=0} = \frac{1}{\sqrt{6}}$$

$$\langle\Sigma^-\pi^0|K^-p\rangle = 0$$

$$\langle\Sigma^-\pi^-|K^-p\rangle = 0.$$

(a) The relative rates are therefore 1:1 for both the $I=0$ and $I=1$ cases.

The $|K^- n\rangle$ system has isospin

$$|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = |1\ -1\rangle$$

and hence,

$$\langle \Sigma^- \pi^0 | K^- n\rangle_{I=1} = -\frac{1}{\sqrt{2}}$$

Thus, when the $I=0$ amplitude dominates, only the $K^- p$ reaction contributes and the relative rates are still 1:1.

When the $I=1$ amplitude dominates

$$P(K^- p \to \Sigma^+ \pi^-) \propto |\tfrac{1}{2}|^2 = \tfrac{1}{4}$$

$$P(K^- p \to \Sigma^- \pi^+) \propto |-\tfrac{1}{2}|^2 = \tfrac{1}{4}$$

$$P(K^- n \to \Sigma^- \pi^0) \propto |-\tfrac{1}{\sqrt{2}}|^2 = \tfrac{1}{2}$$

(b) Thus, the ratio of Σ^+ to Σ^- production cross sections should be 1:3 when K^- are incident on deuterium.

3. The expectation value for the operator $\vec{S}_i \cdot \vec{S}_j$ is given by

$$\langle \vec{S}_1 \cdot \vec{S}_2 \rangle = \frac{1}{2}\left(S(S+1) - s_1(s_1+1) - s_2(s_2+1)\right)$$

$$= \begin{cases} 1/4 & \text{when } S = 1 \\ -3/4 & \text{when } S = 0 \end{cases}$$

Also, $\langle (\vec{S}_1 + \vec{S}_2) \cdot \vec{S}_3 \rangle = \frac{1}{2}\left(S(S+1) - S_{12}(S_{12}+1) - S_3(S_3+1)\right)$

$$= \begin{cases} 0 & \text{when } S_{12} = 0 \\ -1 & \text{when } S_{12} = 1 \text{ and } S = 1/2 \\ 1/2 & \text{when } S_{12} = 1 \text{ and } S = 3/2 \end{cases}$$

For the spin 3/2 baryons $\langle \vec{S}_i \cdot \vec{S}_j \rangle = 1/4$.

Thus, $m_\Delta = 3m_u + \dfrac{3\kappa}{4m_u^2}$

$$m_{\Sigma^{*+}} = 2m_u + m_s + \frac{\kappa}{4m_u^2} + \frac{\kappa}{2m_u m_s}$$

$$= 2(364\text{ MeV}) + (535\text{ MeV}) + \frac{(0.0259\times10^9\text{ MeV}^3)}{4(364\text{ MeV})^2}$$

$$+ \frac{(0.0259\times10^9\text{ MeV}^3)}{2(364\text{ MeV})(535\text{ MeV})}$$

$$= 1378\text{ MeV}$$

This can be compared to the measured mass of 1383 MeV.

For the spin $\frac{1}{2}$ baryons it is useful to first consider cases where there are two identical quarks, like the proton. Because the total wavefunction must be symmetric under the exchange of these quarks, they must be in a $J_z = 0$ state of a $J=1$ multiplet.

$$|p\uparrow\rangle = \frac{1}{\sqrt{2}}\left(u\uparrow u\downarrow + u\downarrow u\uparrow\right)\left(d\uparrow\right) + \text{permutations}$$

Hence, $S_{12} = 1$ and

$$m_p = 3m_u + \frac{\kappa\, \vec{S}_1 \cdot \vec{S}_2}{m_u^2} + \frac{\kappa(\vec{S}_1 + \vec{S}_2)\cdot \vec{S}_3}{m_u^2}$$

$$= 3m_u + \frac{\kappa}{4m_u^2} - \frac{\kappa}{m_u^2}$$

$$= 3m_u - \frac{3\kappa}{4m_u}$$

Likewise, $m_\Sigma = 2m_u + m_s + \frac{\kappa}{4m_u^2} - \frac{\kappa}{m_u m_s}$

and $m_\Xi = 2m_s + m_u + \frac{\kappa}{4m_s^2} - \frac{\kappa}{m_u m_s}$

The Λ has the light quarks in an antisymmetric state so that $S_{12} = 0$.

Hence, $m_\Lambda = 2m_u + m_s - \frac{3\kappa}{4m_u^2}$

mu =	364	MeV			
ms =	535	MeV			
kappa =	2.59E+07	MeV^3			
	m1	m2	m3	Calculated mass (MeV)	Measured mass (MeV)
Δ	364	364	364	1238.61	1232.00
Σ*	364	364	535	1378.37	1384.57
Ξ*	364	535	535	1523.12	1533.40
Ω	535	535	535	1672.87	1672.45
N	364	364	364	945.39	938.92
Σ	364	364	535	1178.87	1193.15
Ξ	535	535	364	1323.62	1318.29
Λ	364	364	535	1116.39	1115.68