PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 8 – French, Chapter 4

Spring 2016 Semester

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Forced Harmonic Motion

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t$$

- Frequency of free oscillations:

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \gamma = \frac{b}{m} \qquad \omega_{free} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

- Driving frequency: ω
- Steady state solution ($t \gg 1/\gamma$):

$$x(t) = A\,cos(\omega t - \delta)$$

- Amplitude of steady-state oscillations:

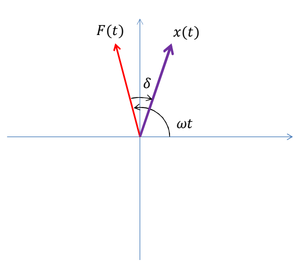
$$A = \frac{F_0/m}{\sqrt{\left((\omega_0)^2 - \omega^2\right)^2 + (\omega\gamma)^2}}$$

- Phase difference:

$$\delta = \tan^{-1}\left(\frac{\omega\gamma}{(\omega_0)^2 - \omega^2}\right)$$

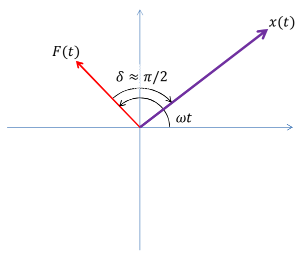
Forced Harmonic Motion

- Phasor diagram: $\omega < \omega_0$



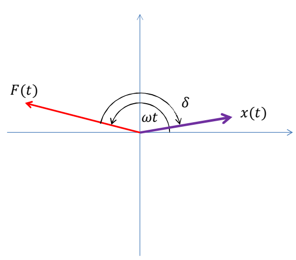
Forced Harmonic Motion

- Phasor diagram: $\omega = \omega_0$



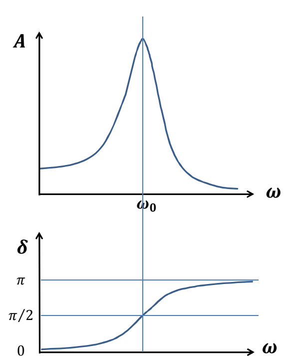
Forced Harmonic Motion

- Phasor diagram: $\omega > \omega_0$



Resonance

- The peak occurs at a frequency that is close to, but not exactly equal to ω_0.
- At resonance, the phase shift is exactly $\delta = \pi/2$.
- The force pushes the mass in the direction it is already moving adding energy to the system.



"Quality Factor"

- Instead of using $\gamma = b/m$ and $\omega_0 = \sqrt{k/m}$, it is convenient to describe the shape of the resonance curve using the variables ω_0 and $Q = \omega_0/\gamma$.
- $Q = \omega_0/\gamma$ is called the "quality factor".
- Written in terms of ω_0 and Q, the amplitude is

$$A = \frac{F_0/m}{\sqrt{\left((\omega_0)^2 - \omega^2\right)^2 + (\omega\omega_0)^2/Q^2}}$$

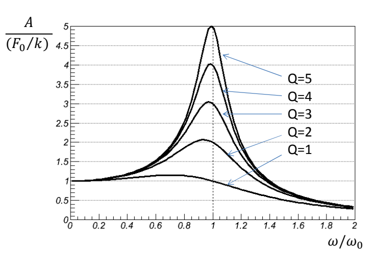
$$= \frac{F_0}{k}\frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

Quality Factor

$$A(\omega) = \frac{F_0}{k}\frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

- Why is this a convenient form?
  - Dimensionless quantities are easier to analyze
  - The scale of the amplitude is determined by F_0/k
  - The shape of the curve is determined by the dimensionless quantities ω/ω_0 and Q

Quality Factor



The normalized height is approximately Q
The maximum occurs when $\omega/\omega_0 \approx 1$
At resonance, the motion is amplified by the factor Q.

Energy

- An oscillator stores energy
- The driving force adds energy to the system
- The damping force dissipates energy
- Instantaneous rate at which energy is added:

$$P = \frac{dW}{dt} = F\frac{dx}{dt} = F\dot{x}$$
$$F(t) = F_0 \cos \omega t$$
$$x(t) = A\, cos(\omega t - \delta)$$
$$\dot{x}(t) = -A\omega \sin(\omega t - \delta)$$
$$P = -F_0 A\omega \cos(\omega t)\sin(\omega t - \delta)$$

- Average rate at which energy is added:

$$\bar{P}(\omega) = \frac{1}{2}F_0 A\omega \sin \delta$$

- Maximal when $\delta = \pi/2$

Energy

$$\bar{P}(\omega) = \frac{1}{2}F_0 A\omega \sin \delta$$

- Some algebra:

$$A(\omega) = \frac{F_0}{k}\frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$
$$\sin \delta = \frac{1/Q}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

- Average power:

$$\bar{P}(\omega) = \frac{(F_0)^2\omega_0}{2kQ}\frac{1}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}$$

Energy

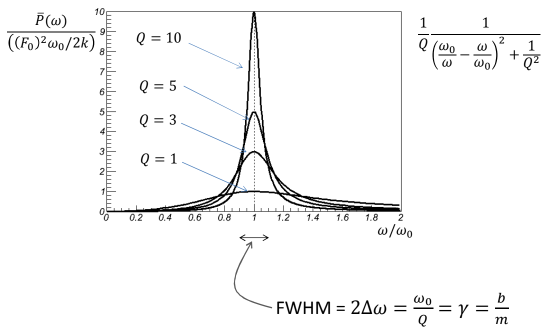
- When $\omega \approx \omega_0$ we can simplify further:

$$\omega = \omega_0 + \Delta\omega$$
$$\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0} \approx \frac{2}{\omega_0}\Delta\omega$$
$$\bar{P}(\omega) = \frac{(F_0)^2(\omega_0/Q)}{2k/(\omega_0)^2}\frac{1}{4(\Delta\omega)^2 + (\omega_0/Q)^2}$$

- What value of $\Delta\omega$ will reduce the peak power by a factor of ½?

$$\frac{1}{4(\Delta\omega)^2 + (\omega_0/Q)^2} = \frac{1}{2}\frac{1}{(\omega_0/Q)^2} \Longrightarrow 2\Delta\omega = \omega_0/Q$$

Power Resonance Shape



Resonance Curves

- General properties:
  - Amplitude at resonance: Static displacement x Q
  - FWHM power bandwidth: $\gamma = \omega_0/Q$
  - When Q is large, a small force at the resonant frequency produces large oscillations
  - Large amplitudes persist only when the frequency of the driving force is near the natural oscillation frequency

Transient Phenomena

- So far, we only considered the form of solutions when t was very large:
  $$x_1(t) = A\,cos(\omega t - \delta)$$
- What is the form of the solution when t is small?
- The solution with no forcing term was
  $$x_2(t) = \boldsymbol{B}e^{-\gamma t/2}\cos\omega_0 t + \boldsymbol{C}\,e^{-\gamma t/2}\sin\omega_0 t$$
- Complete solution:
  $$x(t) = x_1(t) + x_2(t)$$
- Initial conditions determine ***B*** and ***C*** in the complete solution.

Transient Phenomena

- Suppose a mass is already in motion at $t = 0$:

$$x(0) = A_0$$
$$\dot{x}(0) = 0$$

- Suppose that γ is small, so that this motion persists for a long time: ignore the $e^{-\gamma t/2}$ terms

$$x_2(t) = B\cos\omega_0 t$$

- Steady state solution:

$$x_1(t) = A\,cos(\omega t - \delta), \qquad A = \frac{F_0/m}{\sqrt{\left((\omega_0)^2 - \omega^2\right)^2 + (\omega\omega_0)^2/Q^2}}$$

- Complete solution:

$$x(t) = B\cos\omega_0 t + A\,cos(\omega t - \delta)$$

Transient Phenomena

- At $t = 0$, $\dot{x}(t) = 0$:

$$\dot{x}(t) = -B\omega_0\,sin\,\omega_0 t - A\omega\,sin(\omega t - \delta)$$

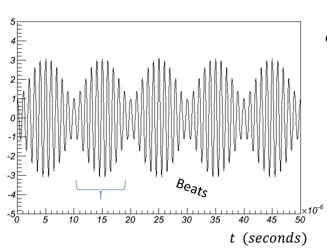
- The phase of the driving force must be 0 or π.
- Amplitude at $t = 0$:

$$A_0 = B + A$$

- A and A_0 are given, so $B = A_0 - A$

$$x(t) = (A_0 - A)\cos\omega_0 t + A\,cos\,\omega t$$
$$= A_0\cos\omega_0 t + A(\cos\omega t - \cos\omega_0 t)$$
$$= A_0\cos\omega_0 t + 2\mathrm{A}\sin(\bar{\omega} t)\sin\left(\frac{1}{2}\Delta\omega t\right)$$

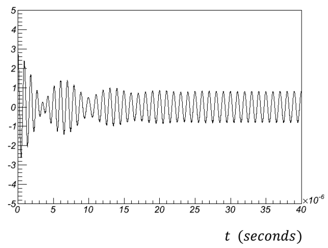
Transient Phenomena



$$\omega_0 = 2\pi \times (1.0\text{ Mhz})$$
$$\omega = 2\pi \times (1.1\text{ Mhz})$$
$$Q = 10^4$$

Transient Phenomena

- Accounting for energy dissipation looks like this:

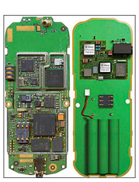


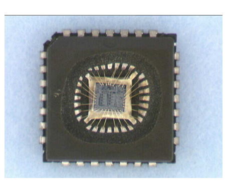
$$\omega_0 = 2\pi \times (1.0 \text{ Mhz})$$
$$\omega = 2\pi \times\ (1.15 \text{ Mhz})$$
$$Q = 20$$
$$\frac{1}{\gamma} = \frac{Q}{\omega_0} = 3.2\ \mu s$$

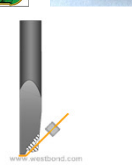
Lifetime of Oscillations

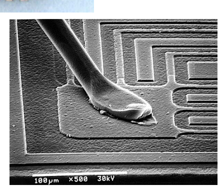
- Amplitude of a damped harmonic oscillator:
$$x(t) = A\, e^{-\gamma t/2} \cos \omega t$$
- Maximum potential energy:
$$U = \frac{1}{2} k\, x^2 \propto e^{-\gamma t}$$
- After time $t = 1/\gamma$, the energy is reduced by the factor $1/e$.
- We call $\tau = 1/\gamma$ the "lifetime" of the oscillator.

Other Resonant Systems

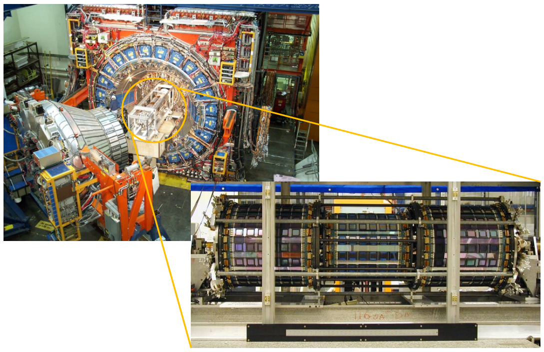








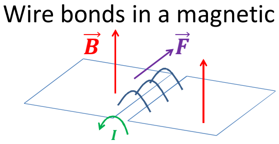
Other Resonant Systems



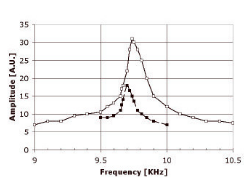
Other Resonant Systems

Wire Bond Resonance

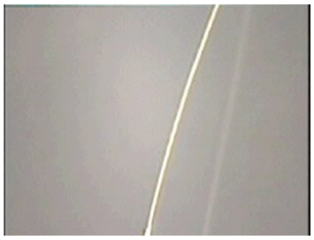
- Wire bonds in a magnetic field:



Lorentz force is $\vec{F} = I \int d\vec{\ell} \times \vec{B}$
The tiny wire is like a spring.
A periodic current produces
the driving force.

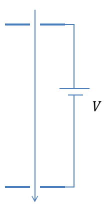


Wire Bond Resonance



Resonance in Nuclear Physics

- A proton accelerated through a potential difference V gains kinetic energy $T = eV$:



Phys. Rev. 75, 246 (1949).

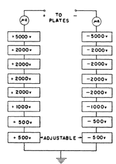


FIG. 1. Block diagram of battery stacks. Adjustable 500-volt boxes were set to any voltage below 500 volts by means of a potentiometer. The polarity of any of the boxes except the adjustable 500-volt box could be selected at will for comparison purposes.

* 5000 volt boxes used Eveready No. 493, 300 volt batteries. All other batteries were of the Burgess XX45, 67½ volt type except for several heavier duty batteries under continuous drain to provide continuous range of adjustments.

** The actual voltage is 504.08 Int. volts and is determined by the resistor divider ratio and the 1.50000 volt setting on the potentiometer.

Resonance in Nuclear Physics

- In quantum mechanics, energy and frequency are proportional:

$$E = \hbar\omega$$

- A given energy corresponds to a driving force with frequency ω.
- When a nucleus resonates at this frequency, the proton energy is easily absorbed.

Nuclear Resonance

ABSOLUTE VOLTAGE DETERMINATION 255

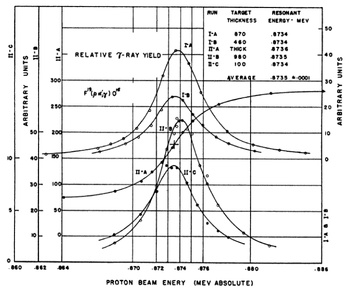
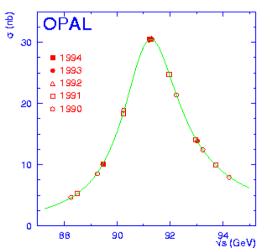
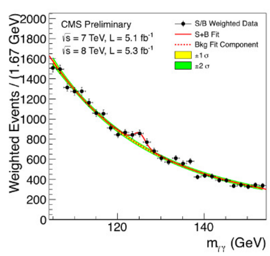


FIG. 7. γ-ray yield curves for both series of measurements of F(pγ) resonance reaction. Yield values are all on same relative scale.

"Lifetime" is defined in terms of the width of the resonance.

Resonance





- Resonances are the main way we observe fundamental particles.