PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 7 – French, Chapter 4

Spring 2016 Semester

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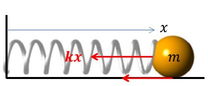
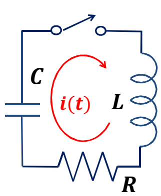
Forced Oscillations and Resonance

This is why you should pay attention.

Simple Harmonic Motion



$$m\ddot{x} + b\dot{x} + kx = 0$$



$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = 0$$

Second-order, *homogeneous*, *linear* differential equations with *constant coefficients*.

Forced Harmonic Motion

- Homogeneous equation:

$$m\ddot{x} + b\dot{x} + kx = 0$$

- Solutions are, for example, $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

$$x(t) = \boldsymbol{A}e^{-\frac{\gamma}{2}t}\sin\omega t + \boldsymbol{B}e^{-\frac{\gamma}{2}t}\cos\omega t$$

- Non-homogeneous equation:

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

- There is an additional time-dependent force that does not depend on x.
- Periodic forcing: $F(t) = F_0\cos\omega t$ where ω and F_0 are the frequency and amplitude of the applied force.

Periodic Forcing

- We are talking about two frequencies:

On Monday, $\boldsymbol{\omega} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ was the natural frequency of oscillations with no external force

Now, $\boldsymbol{\omega}$ is the frequency of the applied for force and is independent of k, m, b.

- What will the solution, $x(t)$, look like?

Periodic Forcing

- For short times, the initial conditions might influence the motion, but this dies away because of the $e^{-\gamma t/2}$ terms (transient motion).
- For longer times, after the transient behavior has died out, the system undergoes "steady-state" motion which should continue indefinitely.
- What is the form of the "steady-state" solution?
- Two scenarios:

$$\omega \ll \omega_0$$
$$\omega \gg \omega_0$$

Periodic Forcing

- When $\omega \ll \omega_0$, the motion has the same ***frequency*** and ***phase*** as the driving force.
- When $\omega \gg \omega_0$, the motion has the same ***frequency*** but is $\mathbf{180°}$ ***out of phase***.
- Maybe the form of the steady-state solution should look something like
$$x(t) = \boldsymbol{A}\cos(\boldsymbol{\omega} t + \boldsymbol{\varphi})$$
- We have to solve for $\boldsymbol{A}$ and $\boldsymbol{\varphi}$.
- These are *not* determined from the initial conditions… this solution only describes the motion for $\gamma t/2 \gg 1$.

Periodic Forcing

- Consider the simpler equation (no viscous damping):
$$m\ddot{x} + kx = F_0 \cos \omega t$$
$$\ddot{x} + (\omega_0)^2 x = \frac{F_0}{m} cos\, \omega t$$
- Proposed solution when $\omega \ll \omega_0$:
$$x(t) = \boldsymbol{A} \cos \omega t$$
- What value of $\boldsymbol{A}$ will satisfy the differential equation?
$$\dot{x}(t) = -\boldsymbol{A}\omega \sin \omega t$$
$$\ddot{x}(t) = -\boldsymbol{A}\omega^2 \cos \omega t$$
- Substitute into the equation:
$$(-\omega^2 + (\omega_0)^2)\boldsymbol{A} \cos \omega t = \frac{F_0}{m} \cos \omega t$$
$$\boldsymbol{A} = \frac{F_0/m}{(\omega_0)^2 - \omega^2}$$

Periodic Forcing

- Consider the simpler equation (no viscous damping):
$$m\ddot{x} + kx = F_0 \cos \omega t$$
$$\ddot{x} + (\omega_0)^2 x = \frac{F_0}{m} cos\, \omega t$$
- Proposed solution when $\omega \gg \omega_0$:
$$x(t) = \boldsymbol{A} \cos(\omega t + \pi) = -\boldsymbol{A} \cos \omega t$$
- What value of $\boldsymbol{A}$ will satisfy the differential equation?
$$\dot{x}(t) = \boldsymbol{A}\omega \sin \omega t$$
$$\ddot{x}(t) = \boldsymbol{A}\omega^2 \cos \omega t$$
- Substitute into the equation:
$$(\omega^2 - (\omega_0)^2)\boldsymbol{A} \cos \omega t = \frac{F_0}{m} \cos \omega t$$
$$\boldsymbol{A} = \frac{F_0/m}{\omega^2 - (\omega_0)^2}$$

Periodic Forcing

- In both cases, the solution is of the form

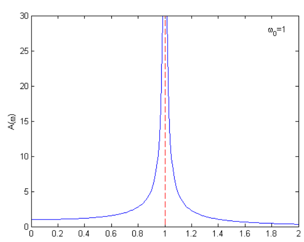
$$x(t) = A\cos(\omega t + \varphi)$$

$$\varphi = \begin{cases} 0 \text{ when } \omega \ll \omega_0 \\ \pi \text{ when } \omega \gg \omega_0 \end{cases}$$

$$A = \frac{F_0/m}{|\omega^2 - (\omega_0)^2|}$$

- What happens when $\omega \approx \omega_0$?
  - Probably nothing good: $A \to \infty$ which is unphysical.

Periodic Forcing



Amplitude gets very large when the frequency of the driving force is close to the natural oscillation frequency.

Periodic Forcing

- Let's derive the form of the solution without any assumptions about ω.
- Assume $x(t)$ is of the form

Real numbers

$$x(t) = Ae^{i(\omega t - \delta)} = Ae^{-i\delta}e^{i\omega t}$$

Just a constant

- Derivatives:

$$\dot{x}(t) = i\omega\, x(t)$$

$$\ddot{x}(t) = -\omega^2\, x(t)$$

- Substitute into the differential equation:

$$m\ddot{x} + b\dot{x} + kx = F_0 e^{i\omega t}$$

Periodic Forcing

- Rewrite the differential equation slightly:

$$\ddot{x} + \gamma\dot{x} + (\omega_0)^2 x = \frac{F_0}{m} e^{i\omega t}$$

- Substitute in the solution:

$$\left[(-\omega^2 + i\omega\gamma + (\omega_0)^2)Ae^{-i\delta}\right]e^{i\omega t} = \frac{F_0}{m} e^{i\omega t}$$

- True for any t provided that

$$A = \frac{e^{i\delta} F_0/m}{(\omega_0)^2 - \omega^2 + i\omega\gamma} = \frac{\left((\omega_0)^2 - \omega^2 - i\omega\gamma\right)e^{i\delta} F_0/m}{\left((\omega_0)^2 - \omega^2\right)^2 + (\omega\gamma)^2}$$

Periodic Forcing

- We said that A was a real number… its magnitude is

$$A = \frac{F_0/m}{\sqrt{\left((\omega_0)^2 - \omega^2\right)^2 + (\omega\gamma)^2}}$$

- The phase, δ, must be

$$\delta = \tan^{-1}\left(\frac{\omega\gamma}{(\omega_0)^2 - \omega^2}\right)$$

- Is this consistent with the expected limits?

Limiting Behavior

- When $\omega \ll \omega_0$ then $\left((\omega_0)^2 - \omega^2\right)^2 \gg (\omega\gamma)^2$

$$A \to \frac{F_0/m}{(\omega_0)^2 - \omega^2}$$

$$\delta = \tan^{-1}(\text{"}small, positive\text{"}) \to 0$$

- When $\omega \gg \omega_0$ then $\left((\omega_0)^2 - \omega^2\right)^2 \gg (\omega\gamma)^2$

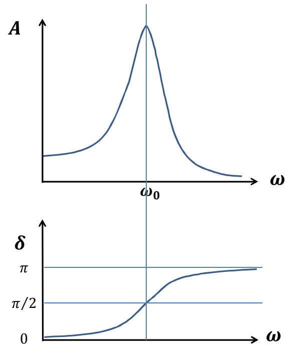
$$A \to \frac{F_0/m}{\omega^2 - (\omega_0)^2}$$

$$\delta = \tan^{-1}(\text{"}small, negative\text{"}) \to \pi$$

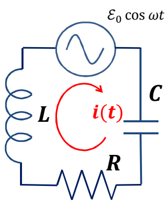
- When $\omega \to 0$ then $A \to F_0/k$.

Resonance

- The peak occurs at a frequency that is close to, but not exactly equal to ω_0.
- At resonance, the phase shift is $\delta = \pi/2$.
- The force pushes the mass in the direction it is already moving adding energy to the system.



Example



$$L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int i(t)\,dt + \mathcal{E}_0\cos\omega t = 0$$

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) - \mathcal{E}_0\omega\sin\omega t = 0$$

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = \mathcal{E}_0\omega\sin\omega t$$

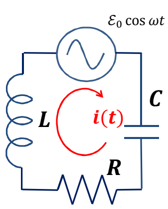
$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i(t) = \frac{\mathcal{E}_0\omega}{L}\sin\omega t$$

$$\frac{d^2 i}{dt^2} + \gamma\frac{di}{dt} + \omega_0^2 i(t) = \frac{\mathcal{E}_0\omega}{L}\sin\omega t$$

Compare with the mechanical analog:

$$\frac{d^2 i}{dt^2} + \gamma\frac{di}{dt} + \omega_0^2 i(t) = \frac{F_0}{m}\sin\omega t$$

Example



$$\frac{d^2 i}{dt^2} + \gamma\frac{di}{dt} + \omega_0^2 i(t) = \frac{\mathcal{E}_0\omega}{L}\sin\omega t$$

This time we can assume the steady state solutions are of the form

$$i(t) = A\ \sin(\omega t - \delta)$$

$$A = \frac{\mathcal{E}_0\omega/L}{\sqrt{\left((\omega_0)^2 - \omega^2\right)^2 + (\omega\gamma)^2}}$$

$$\delta = \tan^{-1}\left(\frac{\omega\gamma}{(\omega_0)^2 - \omega^2}\right)$$