PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 6 – French, Chapter 3

Spring 2016 Semester

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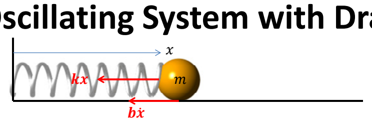
Damping

- Most oscillating physical systems dissipate their energy over time
- We will consider the special cases where the force is a function of velocity

$$F = -b_1 v - b_2 v^2$$

- The drag force is in the opposite direction of the velocity
- Typical of an object moving through a fluid
  - Moving quickly through air: turbulent drag ($b_2 v^2$ is important)
  - Moving slowly through water: viscous drag ($b_1 v$ is important)
- When v is small enough, or b_2 is small enough, only the first term is important.

Oscillating System with Drag



- Newton's second law:

$$m\ddot{x} = -kx - b\dot{x}$$
$$m\ddot{x} + b\dot{x} + kx = 0$$

- What is the solution to this differential equation?
- Let's try this function: $x(t) = Ae^{\alpha t}$
  - then $\dot{x}(t) = \alpha Ae^{\alpha t}$ and $\ddot{x}(t) = \alpha^2 Ae^{\alpha t}$
- Substitute it into the differential equation:

$$(\alpha^2 m + \alpha b + k)Ae^{\alpha t} = 0$$

Oscillating Systems with Drag

$$(\alpha^2 m + \alpha b + k)Ae^{\alpha t} = 0$$

This is true for any value of t only when

$$\alpha^2 m + \alpha b + k = 0$$

Use the quadratic formula:

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4km}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

which we write as

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}$$

Oscillating Systems with Damping

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$x(t) = \boldsymbol{A} e^{-\frac{\gamma}{2}t} e^{t\sqrt{\gamma^2/4 - (\omega_0)^2}} + \boldsymbol{B} e^{-\frac{\gamma}{2}t} e^{-t\sqrt{\gamma^2/4 - (\omega_0)^2}}$$

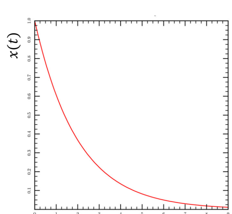
Three cases to consider:

1. $\frac{\gamma^2}{4} - (\omega_0)^2 > 0$ Real roots
2. $\frac{\gamma^2}{4} - (\omega_0)^2 < 0$ Imaginary roots
3. $\frac{\gamma^2}{4} - (\omega_0)^2 = 0$ Degenerate roots

Oscillating Systems with Damping

$$x(t) = Ae^{-\frac{\gamma}{2}t} e^{t\sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}} + Be^{-\frac{\gamma}{2}t} e^{-t\sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}}$$

When $\boldsymbol{\frac{\gamma^2}{4} - (\omega_0)^2 > 0}$ both exponents are real and negative



The mass does not oscillate

It gradually approaches the equilibrium position at $x = 0$.

Oscillating Systems with Damping

$$x(t) = Ae^{-\frac{\gamma}{2}t}e^{t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}} + Be^{-\frac{\gamma}{2}t}e^{-t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}}$$

When $\frac{\gamma^2}{4} - (\omega_0)^2 < 0$ we can write:

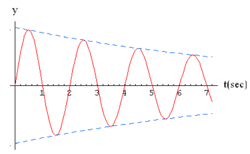
$$x(t) = Ae^{-\frac{\gamma}{2}t}e^{it\sqrt{(\omega_0)^2-\frac{\gamma^2}{4}}} + Be^{-\frac{\gamma}{2}t}e^{-it\sqrt{(\omega_0)^2-\frac{\gamma^2}{4}}}$$

$$= Ce^{-\frac{\gamma}{2}t}\cos(\omega t + \varphi)$$

where $\omega = \sqrt{(\omega_0)^2 - \frac{\gamma^2}{4}}$

Oscillation frequency gradually decreases.

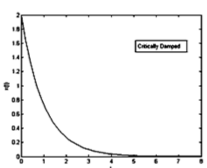
Oscillation frequency is slightly lower than the un-damped oscillator.



Oscillating Systems with Damping

- Third possibility: $\frac{\gamma}{2} - \omega_0 = 0$
- In this case the solution is slightly different:

$$x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$$



fastest return to equilibrium position without oscillating.

Notation

- We just introduced a lot of notation:

$$\omega_0 = \sqrt{k/m}$$

$$\gamma = \frac{b}{m}$$

$$\omega = \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}$$

- None of these express a "fundamental physical law"
- They are just definitions
- If we had defined $\gamma' = \frac{b}{2m}$ then we could write

$$x(t) = Ae^{-\gamma' t}e^{t\sqrt{(\gamma')^2-(\omega_0)^2}} + Be^{-\gamma' t}e^{-t\sqrt{(\gamma')^2-(\omega_0)^2}}$$

- Maybe this is simpler or easier to remember, but it is still rather arbitrary and chosen only for convenience.

Suggestion

- Do not memorize these formulas…
- Instead, memorize this procedure:

$$m\ddot{x} + b\dot{x} + kx = 0$$
$$x(t) = Ae^{\alpha t}$$
$$(\alpha^2 m + \alpha b + k)Ae^{\alpha t} = 0$$
$$\alpha^2 m + \alpha b + k = 0$$
$$\alpha = \frac{-b \pm \sqrt{b^2 - 4km}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

- Then define any new variables you introduce:

$$x(t) = Ce^{-\frac{\gamma}{2}t}\cos(\omega t + \varphi) \text{ (when } \frac{b^2}{4m^2} - \frac{k}{m} < 0)$$

- Recognize that there are three possible forms of the solution.

Example

- Suppose a 1 kg mass oscillates with frequency f and the amplitude of oscillations decreases by a factor of $\frac{1}{2}$ in time T. What differential equation describes the motion?

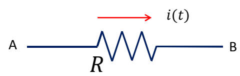
$$m\ddot{x} + b\dot{x} + kx = 0$$
$$m = 1\,kg$$
$$b = \frac{2m\log 2}{T}$$
$$k = m\left(4\pi^2 f^2 + \left(\frac{\log 2}{T}\right)^2\right)$$

RLC Circuits

Review of electricity and magnetism:

- Capacitors store energy in an electric field
- Resistors dissipate energy by heating
- Inductors store energy in a magnetic field
- Voltage
  - Energy per unit charge
  - SI units: *Volt = J/C*
- Current
  - Charge passing a point in a circuit per unit time
  - SI units: *Ampere = C/s*

Resistors

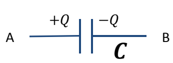


- Current flows from A to B in the direction indicated by the arrow.
- Charges at point B have less energy than at point A because some of their energy was dissipated as heat.
- Potential difference:

$$\Delta V = V_A - V_B = i(t)\, R$$

- Resistance, R, is measured in *ohms* in SI units

Capacitors



- Potential difference:

$$\Delta V = V_A - V_B = \frac{Q}{C}$$

- No current can flow across the capacitor, so any charge that flows onto the plates accumulates there:

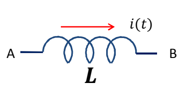
$$Q(t) = Q_0 + \int_0^t i(t)dt$$

- Potential difference:

$$\Delta V = \frac{1}{C}\int_0^t i(t)dt$$

- SI units for capacitance: *farad*

Inductors



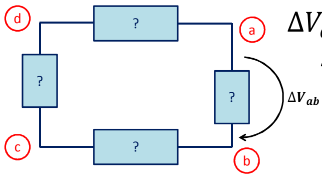
- The inductor will establish a potential difference that opposes any change in current.
  - If $di/dt < 0$ then $V_B > V_A$
    - magnetic field is being converted to energy
  - If $di/dt > 0$ then $V_B < V_A$
    - energy is being stored in the magnetic field

$$\Delta V = V_A - V_B = L\frac{di}{dt}$$

- SI units for inductance: *henry*

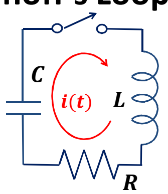
Kirchhoff's Loop Rule

- The sum of the potential differences around a loop in a circuit must equal zero:



$$\Delta V_{ab} + \Delta V_{bc} + \Delta V_{cd} + \Delta V_{da} = 0$$

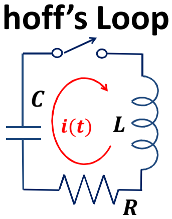
Kirchhoff's Loop Rule



Sum of potential differences:

$$-L\frac{di}{dt}$$

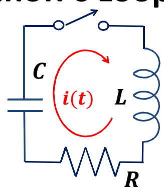
Kirchhoff's Loop Rule



Sum of potential differences:

$$-L\frac{di}{dt} - i(t)R$$

Kirchhoff's Loop Rule



Sum of potential differences:

$$-L\frac{di}{dt} - i(t)R - \frac{1}{C}\left(Q_0 + \int_0^t i(t)dt\right) = 0$$

Initial charge, Q_0, defines the initial conditions.

Kirchhoff's Loop Rule

$$L\frac{di}{dt} + i(t)R + \frac{1}{C}\left(Q_0 + \int_0^t i(t)dt\right) = 0$$

Differentiate once with respect to time:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = 0$$

This is of the same form as the equation for a damped harmonic oscillator:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx(t) = 0$$

Solutions

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = 0$$

Suppose $i(t) = Ae^{\alpha t}$

Then $\frac{di}{dt} = \alpha Ae^{\alpha t}$ and $\frac{d^2i}{dt^2} = \alpha^2 Ae^{\alpha t}$

Substitute into the differential equation:

$$\left(\alpha^2 L + \alpha R + \frac{1}{C}\right)Ae^{\alpha t} = 0$$

True for any t only if $\alpha^2 L + \alpha R + \frac{1}{C} = 0$.

Solutions

$$\alpha^2 L + \alpha R + \frac{1}{C} = 0$$

Roots of the polynomial:

$$\alpha = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Define some new symbols:

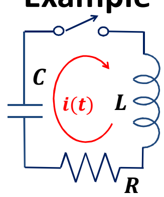
$$\omega_0 = \sqrt{1/_{LC}}$$

$$\gamma = R/L$$

Then the roots can be written

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}$$

Example



- If $L = 2.2\ \mu H$ and $C = 10\ nF$ what is the frequency of oscillations when $R = 0$?
- What is the largest value of R that will still allow the circuit to oscillate?

Example

$$\omega_0 = \sqrt{1/_{LC}} = \sqrt{\frac{1}{(2.2\ \mu H)(0.01\ \mu F)}} = 6.74 \times 10^6\ s^{-1} = 2.15\ MHz$$

$$\gamma = R/L$$

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}$$

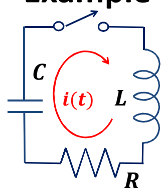
- Critical damping:

$$\frac{\gamma^2}{4} - (\omega_0)^2 = 0$$

$$\gamma = \frac{R}{L} = 2\sqrt{\frac{1}{LC}}$$

$$R = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{2.2\ \mu H}{0.01\ \mu F}} = 29.7\ \Omega$$

Example



$L = 2.2\ \mu H$
$C = 10\ nF$
$R = 2\ \Omega$

- Suppose the initial charge on the capacitor was 10 nC… What voltage is measured across R as a function of time?

Example

- Calculate the discriminant:

$$\frac{R^2}{4L^2} - \frac{1}{LC} = \frac{(2\ \Omega)^2}{4(2.2\ \mu H)^2} - \frac{1}{(2.2\ \mu H)(0.01\ \mu F)} < 0$$

- The circuit will oscillate with frequency:

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = 1.07\ MHz$$

- Time constant:

$$\frac{\gamma}{2} = \frac{R}{2L} = \frac{(2\ \Omega)}{2(2.2\ \mu H)} = 4.55 \times 10^5\ s^{-1}$$

- Current will be:

$$i(t) = i_0 e^{-\gamma t/2} \cos(\omega t + \varphi)$$

Example

$$i(t) = i_0 e^{-\gamma t/2} \cos(\omega t + \varphi)$$

- Initial conditions:
  - $i(0) = 0$ because the inductor produces a potential difference that opposes the change in current.
  - Therefore, $\varphi = \pi/2$
  - Initial potential across capacitor:

$$\Delta V = \frac{Q}{C} = \frac{10\ nC}{10\ nF} = 1\ Volt$$

  - Initial voltage across inductor:

$$\Delta V = L\frac{di}{dt} = 1\ Volt$$

$$\frac{di}{dt} = \frac{1\ V}{2.2\ \mu H} = 4.55 \times 10^5\ A/s = i_0\omega$$

  - $i_0 = \frac{4.55 \times 10^5\ A/s}{6.73 \times 10^6\ /s} = 68\ mA$

Example

- Current in circuit:

$$i(t) = i_0 e^{-\gamma t/2} \sin \omega t$$

where $\gamma/2 = 4.55 \times 10^5\ s^{-1}$ and $i_0 = 68\ mA$

- Potential difference across the resistor:

$$\Delta V = i(t)R$$

$$v(t) = v_0 e^{-\gamma t/2} \sin \omega t$$

where $v_0 = 136\ mV$.