PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 5 – French, Chapter 3

Spring 2016 Semester

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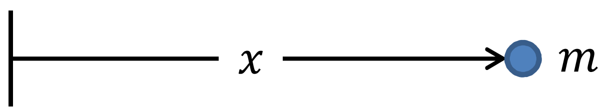
Mechanics Lesson: Circular Motion

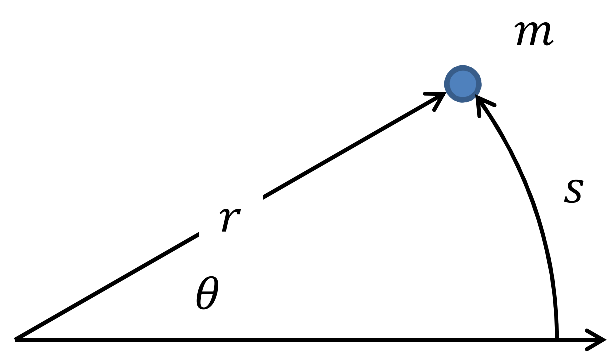
- Linear motion:



  - Mass: m
  - Position: $x(t)$
  - Velocity: $v = \dot{x} = dx/dt$
  - Momentum: $p = m\,\dot{x}$
  - Acceleration: $a = \ddot{x} = d^2x/dt^2$
  - Force: $F = m\,\ddot{x}$
  - Kinetic energy: $T = \frac{1}{2}m\dot{x}^2$

Mechanics Lesson: Circular Motion

- Circular motion:



  - Moment of inertia: $I = mr^2$
  - Angle: $\theta(t)$, arc length: $s(t) = r\,\theta(t)$
  - Angular velocity: $\omega = \dot{\theta}$, linear velocity: $v = r\,\dot{\theta}$
  - Angular momentum: $L = I\,\dot{\theta}$
  - Angular acceleration: $\alpha = \ddot{\theta}$
  - Torque (or "moment"): $N = r\,F = I\,\ddot{\theta}$
  - Kinetic energy: $T = \frac{1}{2}m\dot{s}^2 = \frac{1}{2}mr^2\dot{\theta}^2 = \frac{1}{2}I\,\dot{\theta}^2$

(in 3 dimensions, $\vec{\omega}, \vec{\alpha}, \vec{L}, \vec{N}$ are vectors…)

Free Vibrations of Physical Systems

- Mass + spring system: $m\ddot{x} - kx = 0$
- Stretched elastic material: $m\ddot{x} - \frac{YA}{L}x = 0$
- Floating objects: $m\ddot{x} - \rho g A x = 0$
- Twisted elastic material: $I\ddot{\theta} - \frac{\pi n R^4}{2\ell}\theta = 0$

- What should you have learned from the last lecture?
  - A bunch of formulas?
  - Fundamental physical laws?

Stretched Elastic Material

- Physical concepts:
  - Stuff stretches when you pull on it
  - If it is longer to begin with, it will stretch more, when subjected to the same force
  - Definition: $strain \equiv \Delta l_0 / l_0$
  - But it won't stretch as much if it is thicker
  - Definition: $stress = F/A$
  - Assertion: $strain \propto stress$
    - Limits of applicability? When strain is $\sim few\ \%$

$$F = \frac{YA}{l_0} x$$

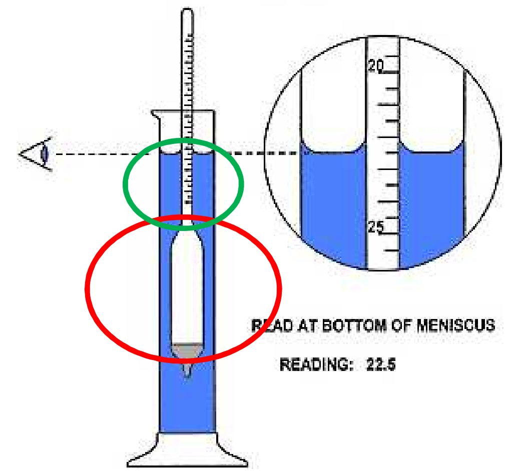
    - This defines the constant of proportionality, Y.

Floating Objects

Warning! People forgot this on the midterm last year!

- Physical concepts:
  - Archimedes’ principle: buoyant force is equal to the weight of displaced liquid.
  - Static equilibrium:

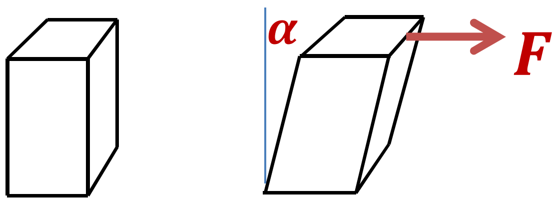
$$\rho g\left(V_0+\pi h\left(\frac{d}{2}\right)^2\right)=mg$$



  - This expression is not worth memorizing…
  - But you should understand what the pieces mean.
  - The details are only specific to this problem

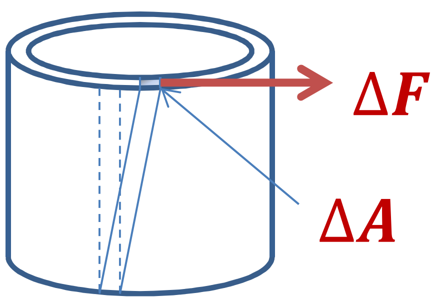
Twisted Elastic Material

- Physical concept:
  - Shear modulus...



$$\frac{F}{A} = n\alpha$$

  - Applied to a specific geometry:



$$M = \frac{\pi n R^4 \theta}{2l}$$

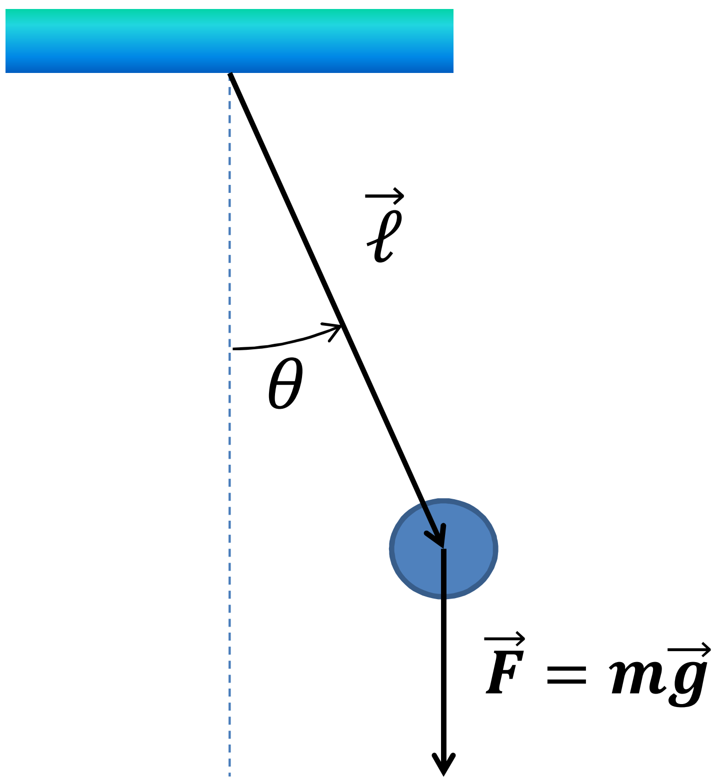
  - Again, the formula is specific to one specific geometry. If the object were rectangular, instead of round, the formula would be different.

More general advice

- Study the examples…
  - Which physical principles are being used?
  - Do you agree with the translation from the physical concepts into algebraic relations?
  - Did the solution require looking at the problem in a different way?
  - Do you understand the geometry?
  - Do you understand the algebra?
  - Could you use the same ideas and techniques to analyze a similar problem?

Almost Linear Systems

- Consider a simple pendulum:



Physical concepts:

- torque produces an angular acceleration.
- definition of torque:

$$\vec{N} = \vec{\ell} \times \vec{F}$$

$$N = -\ell F \sin\theta$$

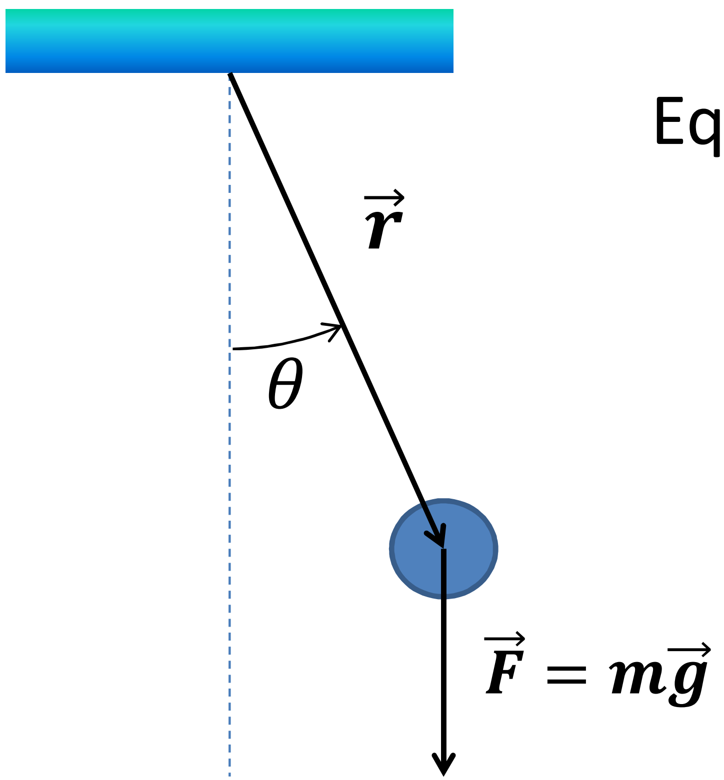
- angular acceleration:

$$N = I\ddot{\theta}$$

- moment of inertia:

$$I = m\ell^2$$

Almost Linear Systems



$$N = I\ddot{\theta}$$

$$N = -\ell mg \sin\theta$$

$$I = m\ell^2$$

Equation of motion:

$$m\ell^2\ddot{\theta} = -mg\ell \sin\theta$$

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

$$\ddot{\theta} + \omega^2 \sin\theta = 0$$

But this is not of the form $\ddot{\theta} + \omega^2\theta = 0$

Almost Linear Systems

$$\ddot{\theta} + \omega^2 \sin\theta = 0$$

- The solution is *not* $\theta(t) = A\cos(\omega t + \varphi)$ but it is close…
- Recall that one way to write $\sin\theta$ is as a power series in θ:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

- When $\theta \ll 1$, $\sin\theta \approx \theta$
- How good is this approximation?
- Suppose we want it to be within 1%

$$\sin(0.3925) - 0.3925 = -0.0100005\ \ldots$$

- In degrees, $22.49° = 0.3925$ radians

Almost Linear Systems

- Provided θ is sufficiently small (ie, $\theta < 22°$),

$$\ddot{\theta} + \omega^2\theta \approx \ddot{\theta} + \omega^2 \sin\theta = 0$$

- The solution is approximately

$$\theta(t) = A\cos(\omega t + \varphi)$$

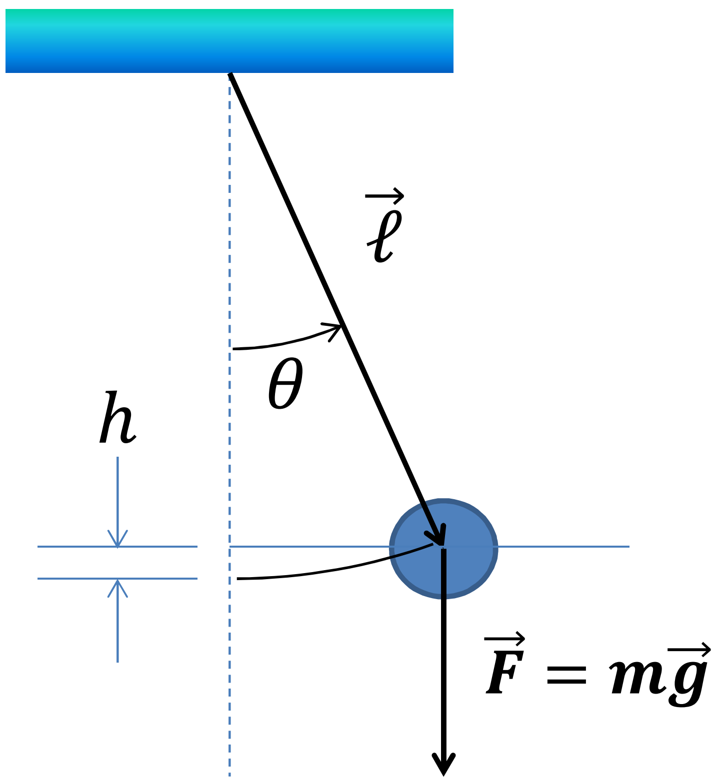
- The frequency is approximately

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$$

- The approximation is better when A is even smaller

Potential Energy Functions

- Same system analyzed using energy:



Kinetic energy:

$$T = \frac{1}{2} I\dot{\theta}^2$$

Potential energy:

$$V = mgh = mg\ell(1 - \cos\theta)$$

Total energy:

$$E = T + V = const.$$

Does this resemble the mass+spring problem?

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E$$

Potential Energy Functions

- Recall that one way to write $\cos\theta$ is as a power series in θ:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

- Energy for a simple pendulum:

$$E = T + V = \frac{1}{2}I\dot{\theta}^2 + mg\ell(1 - \cos\theta)$$

$$T + V \approx \frac{1}{2}I\dot{\theta}^2 + mg\ell\left(1 - \left(1 - \frac{\theta^2}{2!}\right)\right)$$

$$E = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mg\ell\theta^2$$

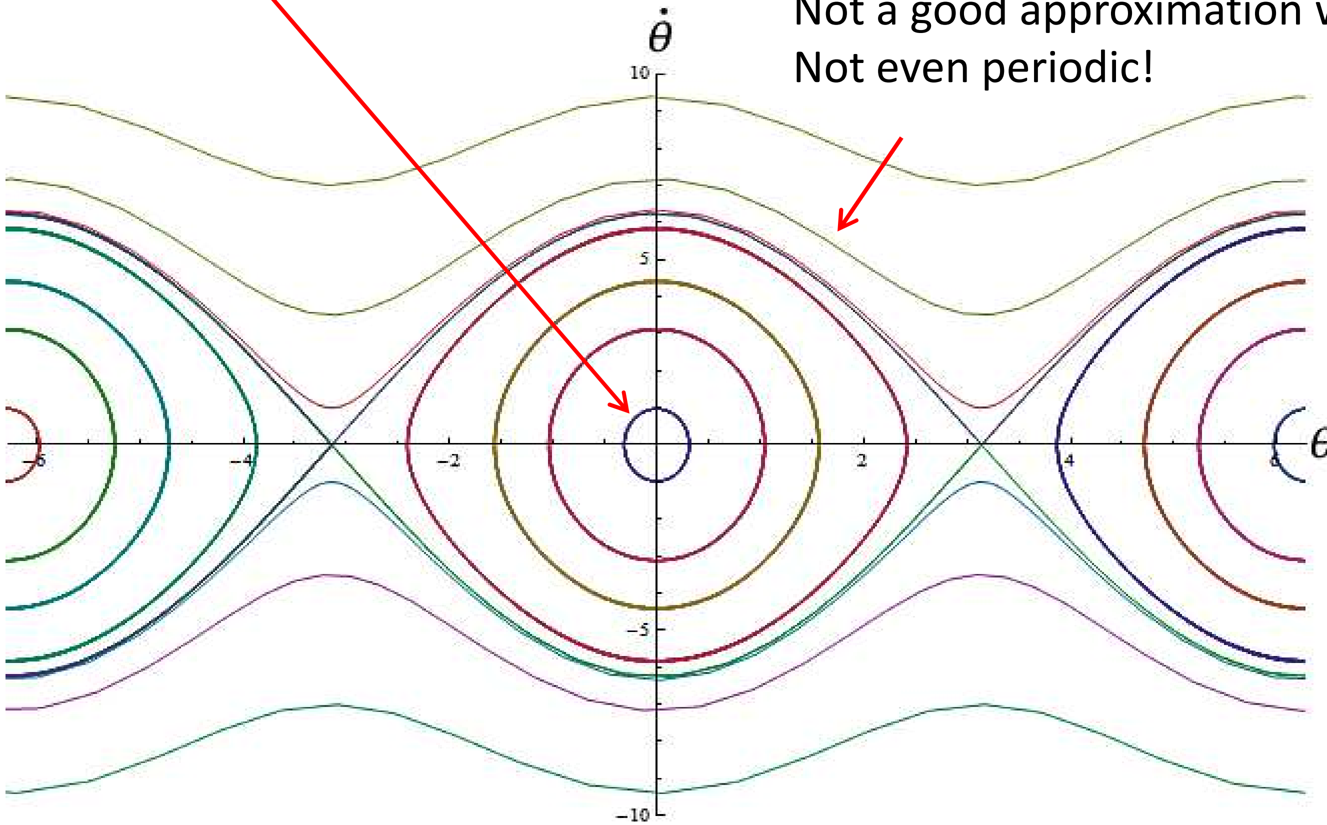
- Now, this is in the same form as for the mass + spring system.
- Interpretation?

Phase Diagram

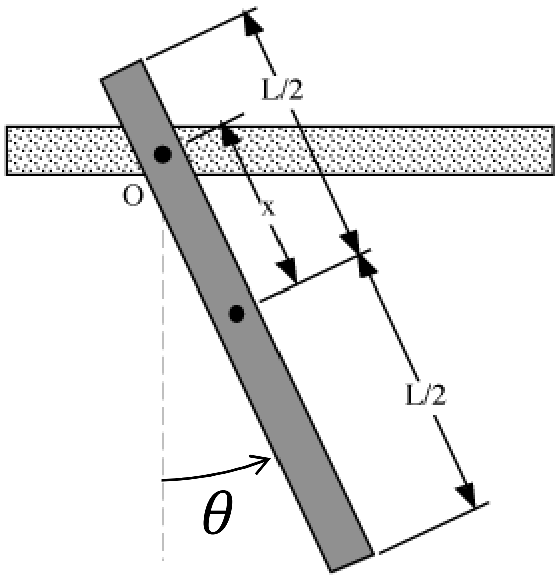
Almost elliptical when E is small.

$$E = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} mg\ell\theta^2 = const$$

Not a good approximation when E is large. Not even periodic!



Physical Pendulum



No new physical concepts – just a different geometry.

$$N = I\ddot{\theta}$$

Gravitational force acts through the center of gravity:

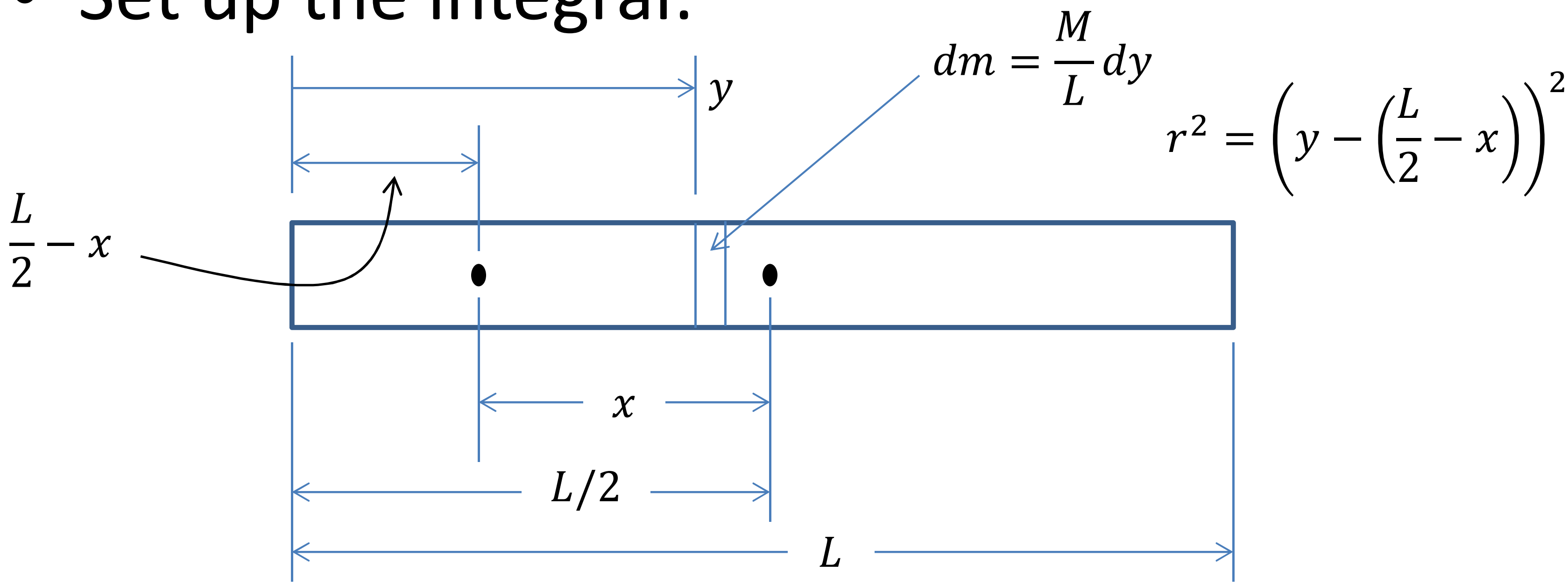
$$N = -Mgx\sin\theta$$

What is the moment of inertia?
Recall that

$$I \equiv \sum_i m_i (r_i)^2 \quad \text{or} \quad I \equiv \int r^2\, dm$$

Moment of Inertia of a Stick

- Set up the integral:



$$I = \frac{M}{L}\int_0^L (y + x - L/2)^2\, dy$$

Let $u = y + x - \frac{L}{2}$

Then $du = dy$

$$I = \frac{M}{L}\int_{x-L/2}^{x+L/2} u^2\, du = \frac{M}{3L}u^3 \Big|_{x-L/2}^{x+L/2}$$

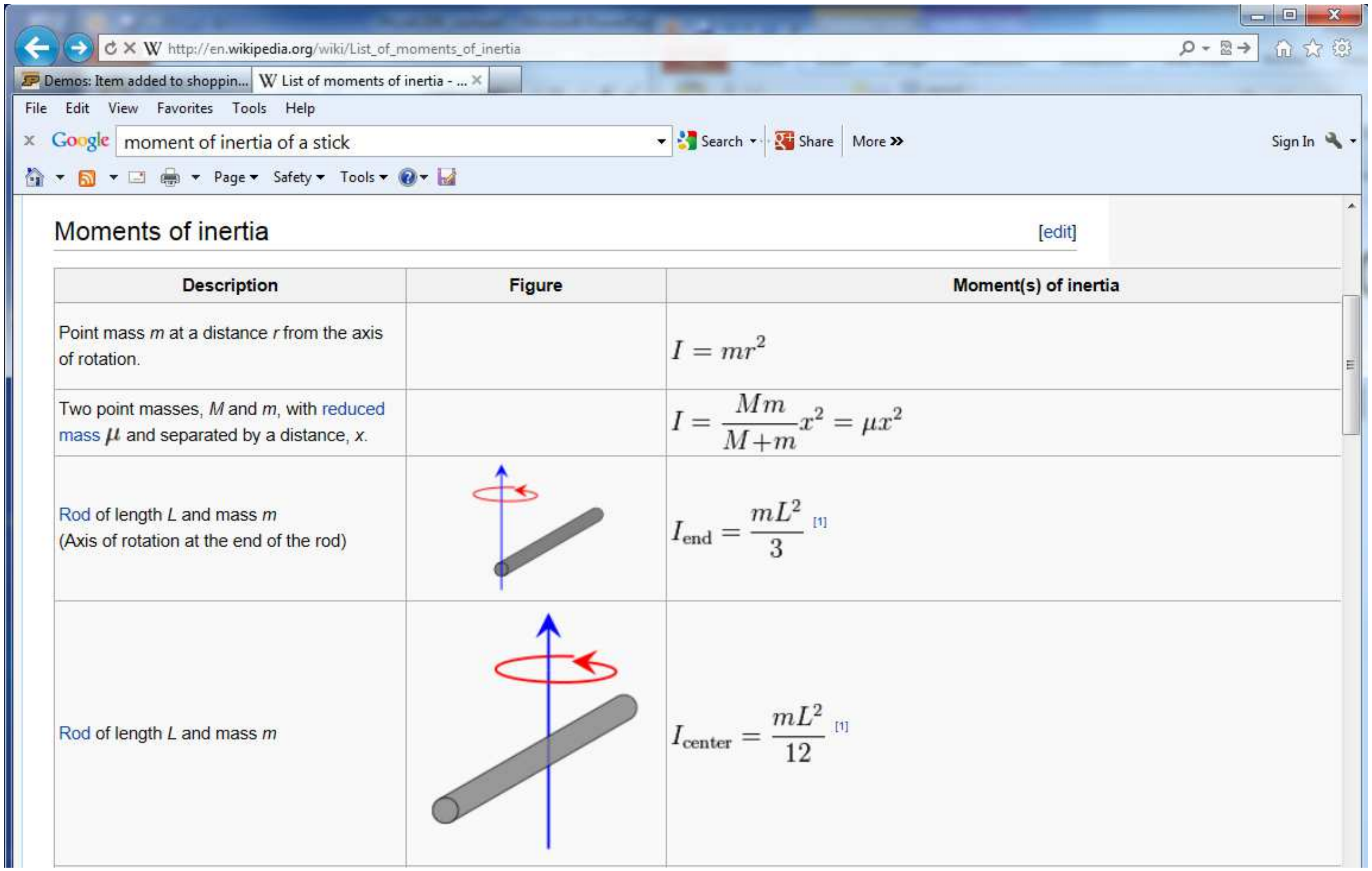
Moment of Inertia of a Stick

$$I = \frac{M}{3L}\left[(x + L/2)^3 - (x - L/2)^3\right]$$
$$= \frac{M}{3L}\left[3x^2 L + L^3/4\right]$$
$$= M\left[x^2 + \frac{L^2}{12}\right]$$

Check the limiting cases:

$x = 0 \rightarrow I = \frac{ML^2}{12}$ $\quad$ $x = \frac{L}{2} \rightarrow I = \frac{ML^2}{3}$

Moment of Inertia of a Stick



Moments of inertia

[edit]

Description	Figure	Moment(s) of inertia
Point mass *m* at a distance *r* from the axis of rotation.		$I = mr^2$
Two point masses, *M* and *m*, with reduced mass μ and separated by a distance, *x*.		$I = \frac{Mm}{M+m}x^2 = \mu x^2$
Rod of length *L* and mass *m* (Axis of rotation at the end of the rod)		$I_{\mathrm{end}} = \frac{mL^2}{3}$ [1]
Rod of length *L* and mass *m*		$I_{\mathrm{center}} = \frac{mL^2}{12}$ [1]

Physical Pendulum

- Equation of motion:

$$I\ddot{\theta} + Mgx\sin\theta = 0$$
$$\ddot{\theta} + \omega^2\theta \approx 0$$

where $\omega = \sqrt{\frac{Mgx}{I}} = \sqrt{\frac{gx}{x^2+\frac{L^2}{12}}}$

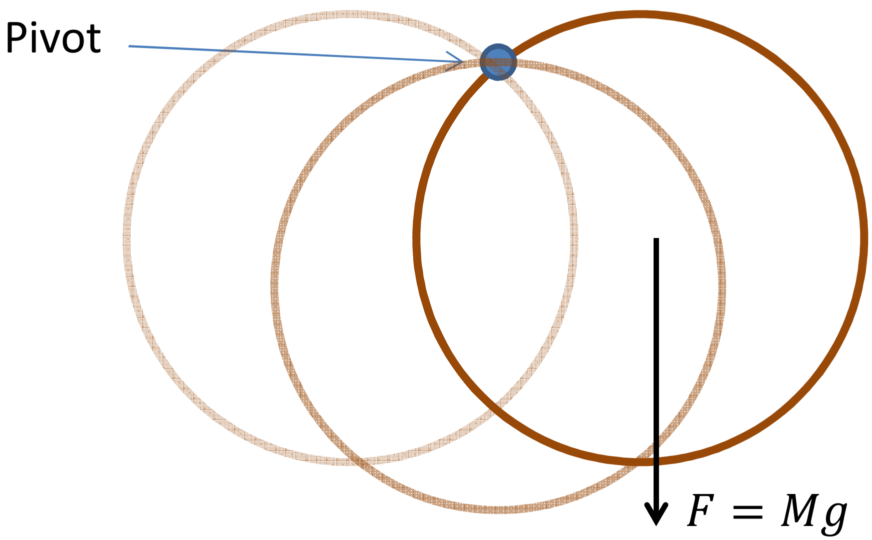
- When $x = L/2$ (suspended from one end)

$$\omega = \sqrt{\frac{3g}{2\ell}}$$

(same frequency as a simple pendulum with 2/3 the length)

One More Physical Pendulum

- The "ring pendulum":



$$I\ddot{\theta} + MgR\sin\theta = 0$$

$$I = 2MR^2$$

$$\omega \approx \sqrt{\frac{MgR}{2MR^2}} = \sqrt{\frac{g}{2R}}$$

You should recognize that the problem is the same as in the case of the stick. The only difference is the moment of inertia.