PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 4 – French, Chapter 3

Spring 2016 Semester

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Energy Considerations

- The force in Hooke's law is

$$F(x) = -kx$$

- Potential energy can be used to describe conservative forces:

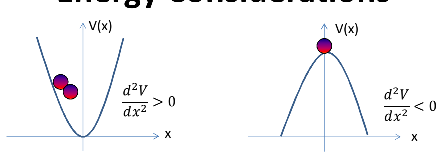
$$\vec{F} = -\nabla V(\vec{x})$$

$$F = -\frac{dV}{dx}$$

- The force vanishes when $dV/dx = 0$
- Local *minimum* when

$$\frac{d^2V}{dx^2} > 0$$

Energy Considerations



- The system will oscillate about a stable equilibrium point.
- If the minimum is parabolic, then the spring constant is

$$k = \frac{d^2V}{dx^2}$$

- Potential energy function for a spring:

$$V(x) = \frac{1}{2}k\,x^2$$

Energy Considerations

- Kinetic energy:

$$T = \frac{1}{2} m\, \dot{x}^2$$

- Total energy:

$$E = T + V = \frac{1}{2} m\, \dot{x}^2 + \frac{1}{2} k\, x^2$$

- Total energy is conserved:

$$\frac{dE}{dt} = 0$$

Simple Harmonic Motion

- Start from Newton's second law:

$$m\ddot{x} + kx = 0$$

- Multiply by $\dot{x}$:

$$m\dot{x}\ddot{x} + k\dot{x}x = 0$$

- Notice that

$$\frac{d}{dt}\dot{x}^2 = 2\dot{x}\ddot{x}$$

$$\frac{d}{dt}x^2 = 2x\dot{x}$$

- So we can write

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$$

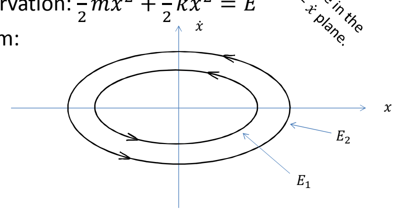
- Which implies that

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E = const.$$

Simple Harmonic Motion

$$x(t) = A\ \cos(\omega t + \varphi)$$
$$\dot{x}(t) = -A\,\omega \sin(\omega t + \varphi)$$

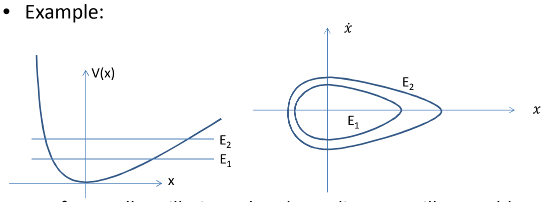
- Energy conservation: $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E$
- Phase diagram:



- The energy conservation relation can tell us a lot about the motion even when we can't solve for $x(t)$.

Phase Diagrams

- Phase diagrams are useful for describing the motion even when we can't solve for $x(t)$ exactly.
- Example:



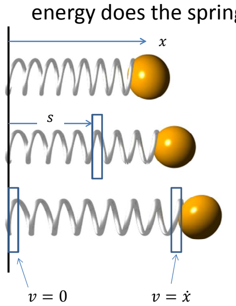
- But for small oscillations the phase diagram will resemble an ellipse.
- Study Assignment #2 from 2014 for an example.

A More Realistic Model

- So far we considered a mass attached to a spring.
- The spring was assumed to be massless.
- What if the spring has a finite mass M?

Physical Spring

- When the mass is in motion, how much kinetic energy does the spring have?



1. The spring has a total length x and total mass M
2. The velocity of the fixed end of the spring is always zero
3. The velocity of the moving end of the spring is given by $\dot{x}$
4. At a distance s from the fixed end, the velocity will be
$$v = \frac{s}{x}\dot{x}$$
5. The mass of an element of length ds will be
$$dM = \frac{M}{x}ds$$

Physical Spring

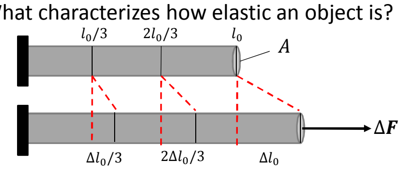
- Kinetic energy of one element of the spring:
$$dT = \frac{1}{2}v^2 dM = \frac{1}{2}\left(\frac{s}{x}\dot{x}\right)^2 \frac{M}{x} ds$$
- We get the total kinetic energy by integrating over the length of the spring:
$$T_{spring} = \frac{M}{2x^3}(\dot{x})^2 \int_0^x s^2 ds = \frac{M}{6x^3}(\dot{x})^2 s^3 \Big|_0^x$$
$$= \frac{M}{6}(\dot{x})^2$$
- Total kinetic energy is $T = T_{mass} + T_{spring}$

Physical Spring

- Total kinetic energy:
$$T = \frac{1}{2}m(\dot{x})^2 + \frac{1}{6}M(\dot{x})^2 = \frac{1}{2}\left(m + \frac{M}{3}\right)(\dot{x})^2$$
- Potential energy:
$$V = \frac{1}{2}kx^2$$
- Total energy:
$$E = T + V = \frac{1}{2}\left(m + \frac{M}{3}\right)(\dot{x})^2 + \frac{1}{2}kx^2$$
- We know that when $E = \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}kx^2$ the frequency is $\omega = \sqrt{k/m}$
- Therefore, the oscillation frequency of the physical spring must be
$$\omega = \sqrt{\frac{k}{m + M/3}}$$

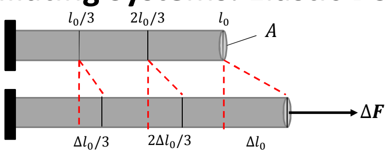
Oscillating Systems: Elastic Bodies

- Rigid bodies are usually elastic although we may not normally notice.
- What characterizes how elastic an object is?



- The extension under the force ΔF is proportional to the original length, l_0.
- Constant of proportionality: $strain \equiv \Delta l_0/l_0$

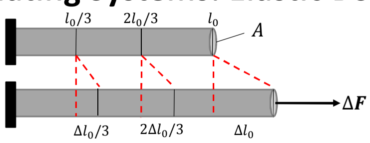
Oscillating Systems: Elastic Bodies



- The same deformation would result if ΔF were increased provided A also increased by the same amount.
- Stress is defined: $stress = \Delta F/A$
- When the strain is small (eg, $\Delta l_0/l_0 < 1\%$), the stress is proportional to the strain:

$$stress \propto strain$$

Oscillating Systems: Elastic Bodies



$$stress \propto strain$$

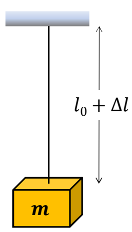
- Constant of proportionality is called Young's modulus

$$\frac{\Delta F}{A} = Y\frac{\Delta l_0}{l_0}$$

- Newton's third law: when the material is stretched by a distance x, the material will exert a reaction force

$$F = -\frac{YAx}{l_0} = -kx \text{ where } k = YA/l_0.$$

Example



- Steel has $Y = 20 \times 10^{10}\ N/m^2$
- Suppose that $m = 1\ kg,\ l_0 = 2\ m$ and has a diameter of $d = 0.5\ mm$ (24 AWG)
- Cross sectional area is

$$A = \pi\left(\frac{d}{2}\right)^2$$

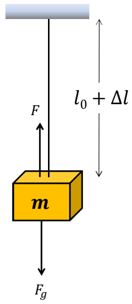
- Restoring force:

$$F = -\frac{YA\Delta l}{l_0} = -\frac{\pi}{4}\frac{Yd^2}{l_0}\Delta l = -k\Delta l$$

$$k = \frac{\pi \cdot (20 \times 10^{10}\ N/m^2) \cdot (0.0005\ m)^2}{4 \cdot (2\ m)}$$

$$= 1.96 \times 10^4\ N/m$$

Example

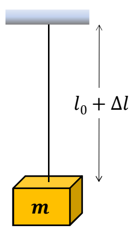


- How much will the wire stretch under the weight of the mass, m?

$$F_g = mg = k\Delta l$$

$$\Delta l = \frac{mg}{k} = \frac{(1\ kg)\cdot(9.81\ N/kg)}{1.96\times 10^4\ N/m} = 5.00\times 10^{-4}\ m$$

Example



- Newton's second law:

$$m\frac{d^2}{dt^2}\Delta l = -k\Delta l$$

$$\frac{d^2}{dt^2}\Delta l + \frac{k}{m}\Delta l = 0$$

$$\frac{d^2}{dt^2}\Delta l + \omega^2\Delta l = 0$$

- Solutions can be written

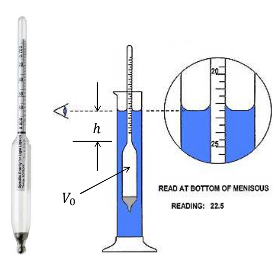
$$\Delta l(t) = A\ \cos(\omega t + \varphi)$$

- Oscillation frequency is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{k/m} = \frac{1}{2\pi}\sqrt{\frac{1.96\times 10^4\ N/m^2}{1\ kg}} = 22.3\ Hz$$

Floating Objects

- Hygrometer: measures density of liquids



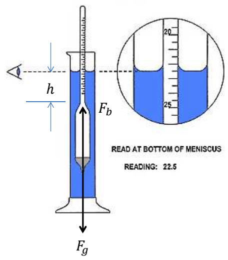
Archimedes' principle: Buoyant force is equal to the weight of the volume of liquid displaced.

If the stem has a diameter of d then the displaced volume is

$$V = V_0 + \pi h\left(\frac{d}{2}\right)^2$$

$$F_b = \rho g\left(V_0 + \pi h\left(\frac{d}{2}\right)^2\right)$$

Floating Objects



When in static equilibrium,

$$F_b = F_g$$

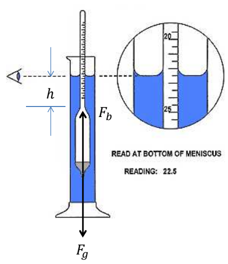
$$\rho g \left(V_0 + \pi h \left(\frac{d}{2} \right)^2 \right) = mg$$

$$h = \frac{m/\rho - V_0}{\pi d^2/4}$$

When the hydrometer is displaced by an additional distance Δh, the net force is

$$F = -\pi \rho g \left(\frac{d}{2} \right)^2 \Delta h$$

Floating Objects

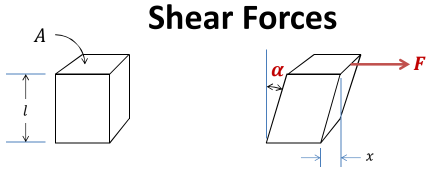


$$m \frac{d^2}{dt^2} \Delta h = -\pi \rho g \left(\frac{d}{2} \right)^2 \Delta h$$

$$\frac{d^2}{dt^2} \Delta h + \omega^2 \Delta h = 0$$

where $\omega = \frac{d}{2} \sqrt{\frac{\pi \rho g}{m}}$

Shear Forces



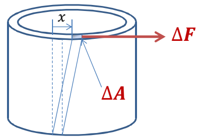
- Angle α is proportional to F and inversely proportional to A:

$$\frac{F}{A} = n\alpha \approx n \frac{x}{l}$$

- The constant of proportionality is called the *shear modulus*, denoted n.
- For example, steel has $n = 8 \times 10^{10}\ N/m^2$

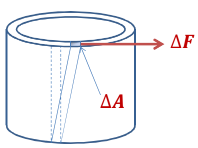
Shear Forces

- Torsion of a thin-walled tube of radius r and length l twisted through an angle θ:



- Angle of deflection: $\frac{x}{l} = \frac{r\theta}{l}$
- Shear force: $\Delta F = \frac{n r \theta \Delta A}{l}$

Shear Forces



- Differential element of torque:
$$\Delta M = r\Delta F$$
- Differential element of area:
$$\Delta A = r\Delta r \Delta\varphi$$
- Integrate around the circle…

$$M = \int dM = r\int dF = \frac{nr^2\theta}{l}\int dA = \frac{nr^2\theta}{l}\int_0^{2\pi}(r\Delta r)d\varphi = \frac{2\pi n r^3 \Delta r\, \theta}{l}$$

- Total torque on a solid cylinder of radius R: integrate over r from 0 to R.

$$M = \frac{2\pi n\theta}{l}\int_0^R r^3 dr = \frac{\pi n R^4\theta}{2l}$$

Torsion Pendulum

- Suppose an object with moment of inertia $I = 0.00167\ kg \cdot m^2$ is suspended from a steel wire of length $\ell = 2\ m$ with a diameter of $d = 0.5\ mm$ (24 AWG).

$$I\ddot{\theta} = -\frac{\pi n R^4 \theta}{2\ell}$$
$$\ddot{\theta} + \omega^2\theta = 0$$
$$\text{where } \omega = \sqrt{\frac{\pi n R^4}{2I\ell}} = \sqrt{\frac{\pi n d^4}{32 I\ell}}$$

- Frequency of oscillation:

$$f = \frac{1}{2\pi}\sqrt{\frac{\pi \cdot (8 \times 10^{10}\ N/m^2)\cdot(0.0005\ m)^4}{32\cdot(0.00167\ kg\cdot m^2)\cdot(2\ m)}} = 0.061\ \text{Hz}$$