Purdue University Department of Physics

Physics 42200

Waves & Oscillations

Lecture 37 – Interference

Spring 2016 Semester

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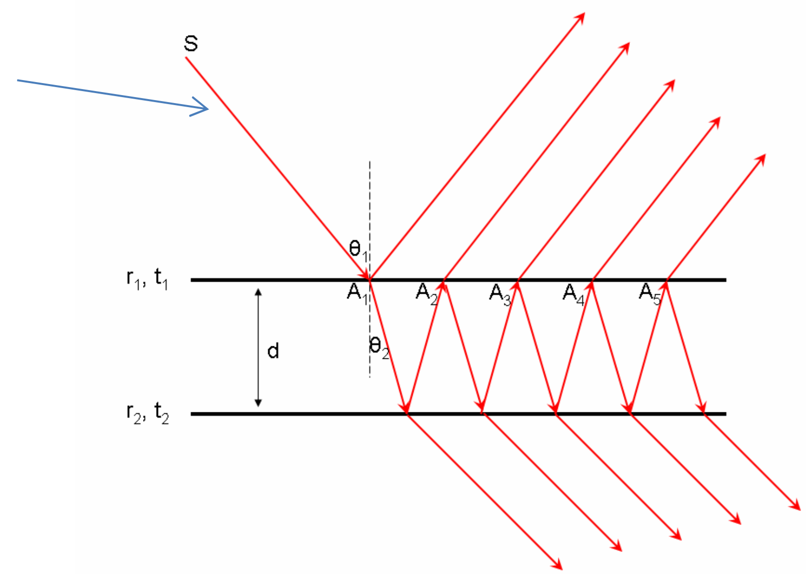
Multiple Beam Interference

- In many situations, a coherent beam can interfere with itself multiple times
- Consider a beam incident on a thin film
  - Some component of the light will be reflected at each surface and some will be transmitted

Incident light

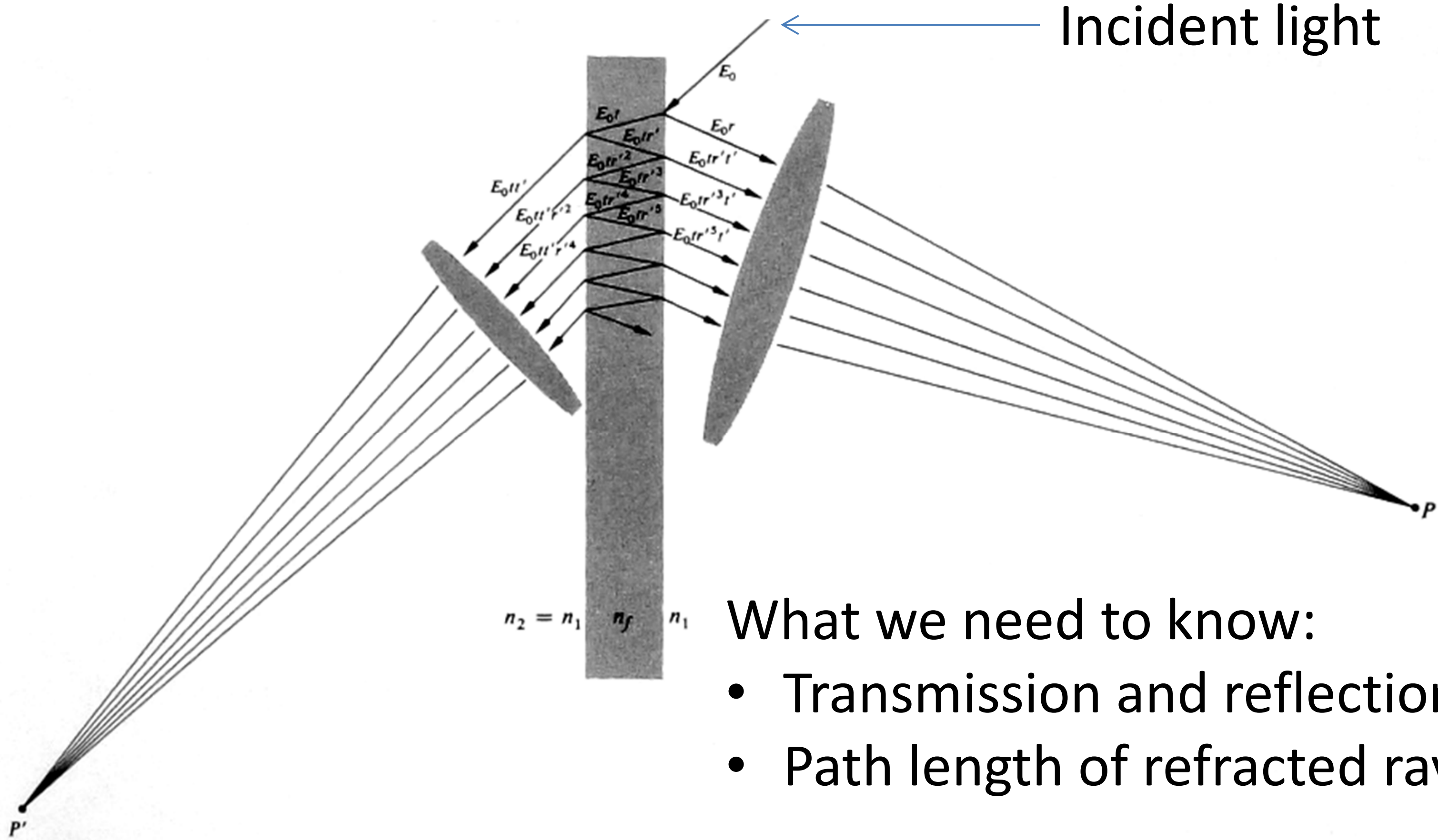


Each transmitted beam will have a different phase relative to the adjacent beams.

What is the total intensity of the reflected light?

Multiple Beam Interference

- All transmitted and reflected rays will be parallel
- They can be focused onto points P and P' by lenses:

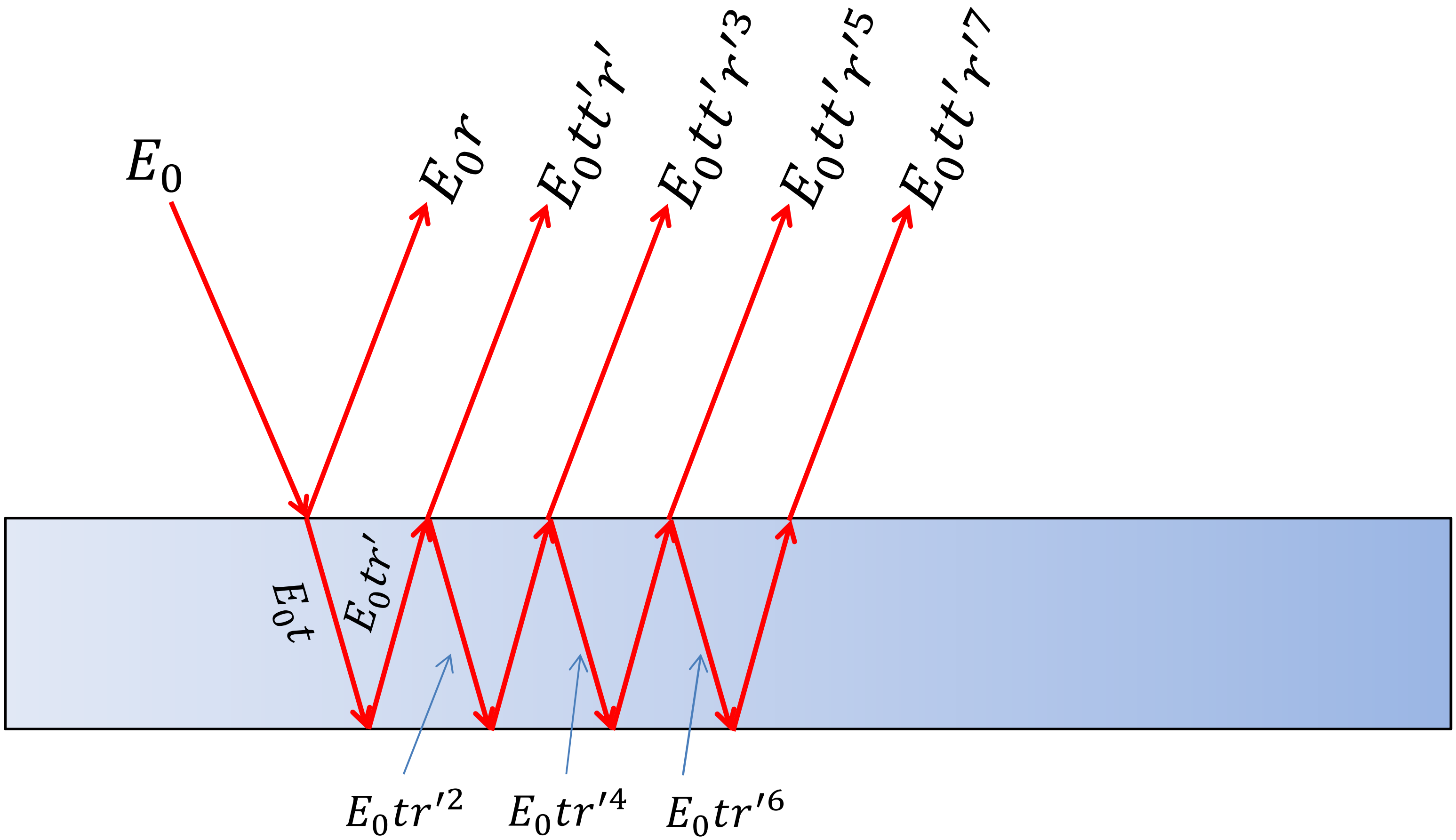


What we need to know:

- Transmission and reflection coefficients
- Path length of refracted rays in the film

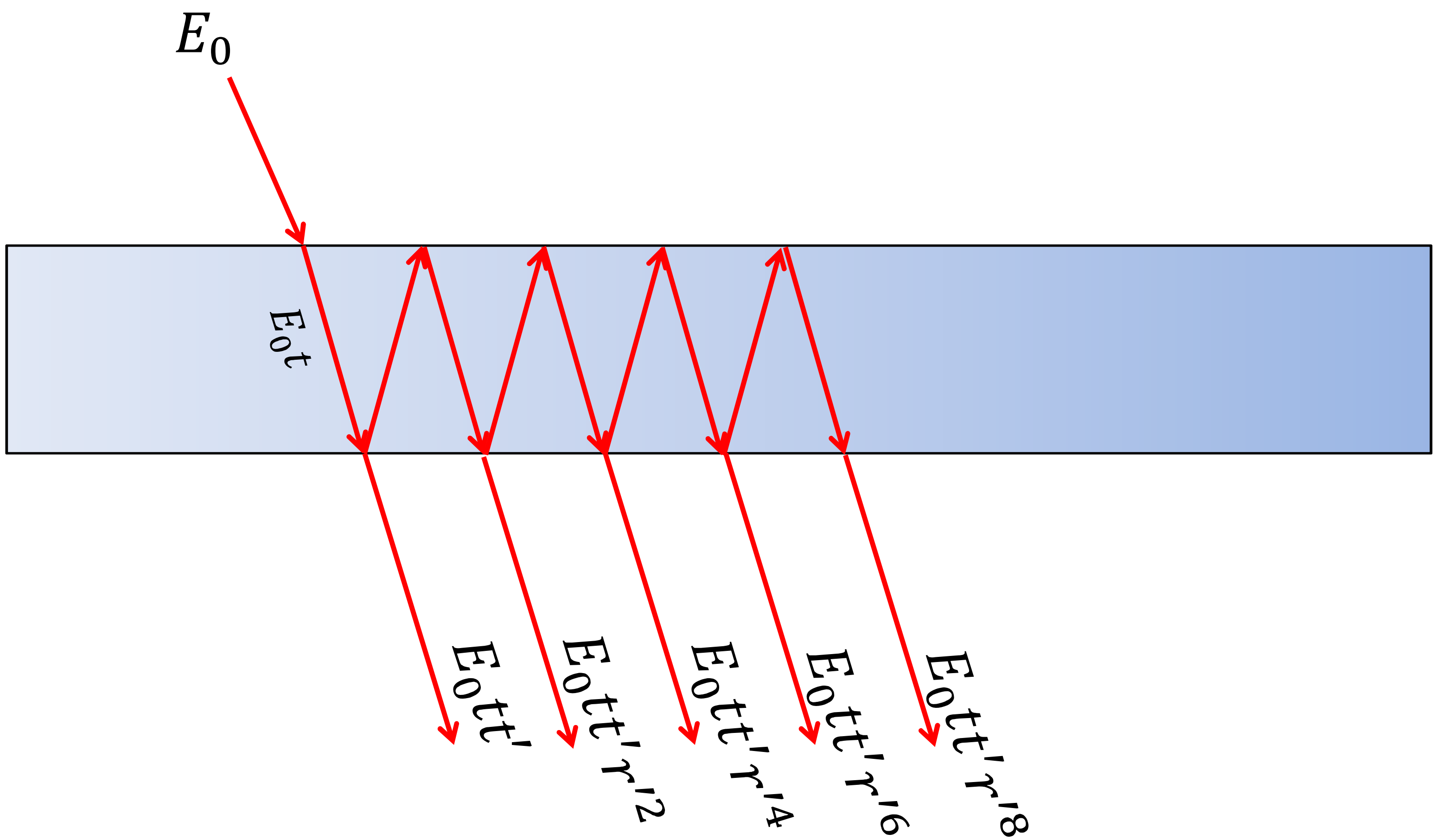
Multiple Beam Interference

- Reflection coefficients: r and r'
- Transmission coefficients: t and t'



Multiple Beam Interference

- Reflection coefficients: r and r'
- Transmission coefficients: t and t'



Multiple Beam Interference

- The additional phase in the film is always the same:

$$\delta = \frac{2nkd}{\cos\theta_t}$$

- If the initial phase is zero, then

$$E_{1r} = E_0 r e^{i\omega t}$$
$$E_{2r} = E_0 t t' r' e^{i(\omega t - \delta)}$$
$$E_{3r} = E_0 t t' r'^3 e^{i(\omega t - 2\delta)}$$
$$E_{4r} = E_0 t t' r'^5 e^{i(\omega t - 3\delta)}$$
$$\cdots$$

- In general:

$$E_{Nr} = E_0 e^{i\omega t} t t' \, r'^{2N-3} e^{-i(N-1)\delta}$$
$$= E_0 e^{i\omega t} t t' r' e^{-i\delta} \left(r'^2 e^{-i\delta}\right)^{N-2}$$

Multiple Beam Interference

- The total electric field on one side of the film:

$$E_r = E_0 e^{i\omega t} r + E_0 e^{i\omega t} tt'r'e^{-i\delta} \times \left[1 + r'^2 e^{-i\delta} + \left(r'^2 e^{-i\delta}\right)^2 + \left(r'^2 e^{-i\delta}\right)^3 + \cdots\right]$$

- This is in infinite sum of the form:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{(when } |z| < 1\text{)}$$

- Total electric field:

$$E_r = E_0 e^{i\omega t}\left[r + \frac{tt'r'e^{-i\delta}}{1 - r'^2 e^{-i\delta}}\right]$$

Multiple Beam Interferometry

- Simplifications:

$$r' = -r$$
$$tt' = 1 - r^2$$

- Total electric field:

$$E_r = E_0 e^{i\omega t} r \left[1 - \frac{(1 - r^2) e^{-i\delta}}{1 - r^2 e^{-i\delta}} \right]$$
$$= E_0 e^{i\omega t} r \left[\frac{1 - r^2 e^{-i\delta} - e^{-i\delta} + r^2 e^{-i\delta}}{1 - r^2 e^{-i\delta}} \right]$$
$$= E_0 e^{i\omega t} r \left[\frac{1 - e^{-i\delta}}{1 - r^2 e^{-i\delta}} \right]$$

Multiple Beam Interferometry

- The intensity of the light is $I_r \propto |E_r|^2$

$$I_r = I_0 \left\{ r \left[\frac{1 - e^{-i\delta}}{1 - r^2 e^{-i\delta}} \right] \right\}^* \left\{ r \left[\frac{1 - e^{-i\delta}}{1 - r^2 e^{-i\delta}} \right] \right\}$$

$$= I_0 r^2 \frac{(1 - e^{i\delta})(1 - e^{-i\delta})}{(1 - r^2 e^{i\delta})(1 - r^2 e^{-i\delta})}$$

$$= I_0 \frac{2r^2(1 - \cos\delta)}{(1 + r^4) - 2r^2 \cos\delta}$$

- The intensity of the transmitted light is $I_t \propto |E_t|^2$

$$I_t = I_0 \frac{1 - r^2}{(1 + r^4) - 2r^2 \cos\delta}$$

Multiple Beam Interferometry

- One more identity will clean this up a bit:

$$\cos\delta = 1 - 2\sin^2(\delta/2)$$

- Reflected intensity:

$$I_r = I_0 \, \frac{F\sin^2(\delta/2)}{1 + F\sin^2(\delta/2)}$$

- Transmitted intensity:

$$I_t = I_0 \, \frac{1}{1 + F\sin^2(\delta/2)}$$

- The parameter $F = \left(\frac{2r}{1-r^2}\right)^2$ is called the ***coefficient of finesse***
- Notice that $I_0 = I_r + I_t$
  - We assumed that no energy was lost in the film

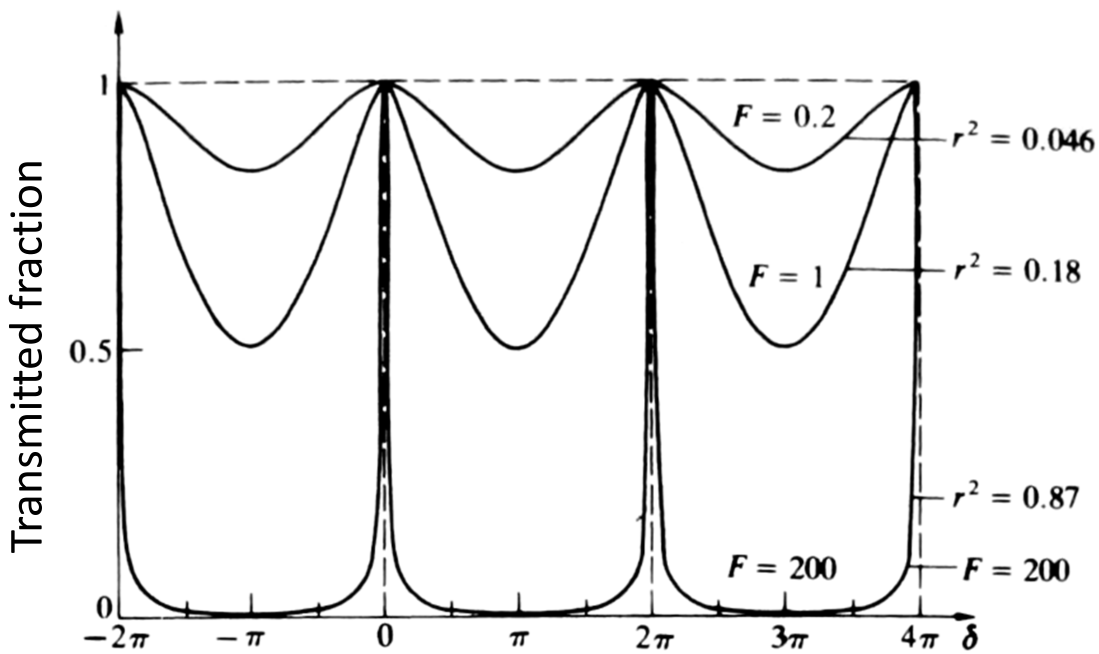
Multiple Beam Interferometry

The function

$$\mathcal{A}(\theta) = \frac{1}{1 + F\sin^2(\delta/2)}$$

is called the Airy function.

Remember that δ is a function of the angle of incidence, θ.



Multiple Beam Interferometry

- In practice, some fraction of the light will be absorbed
- Absorptance, A, is defined by:

$$T + R + A = 1$$

- This modifies the transmitted intensity:

$$I_t = I_0 \left[1 - \frac{A}{1-R}\right]^2 \mathcal{A}(\theta)$$

- Example: silver film, 50 nm thick, deposited on glass

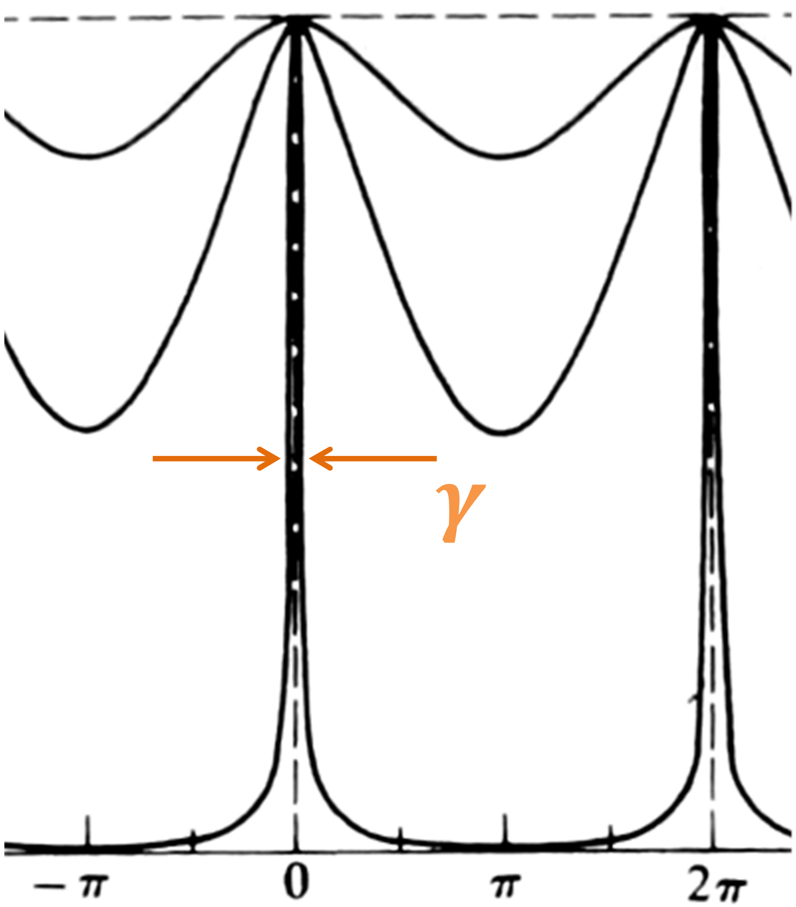
$$R = 0.94, T = 0.01, A = 0.05$$

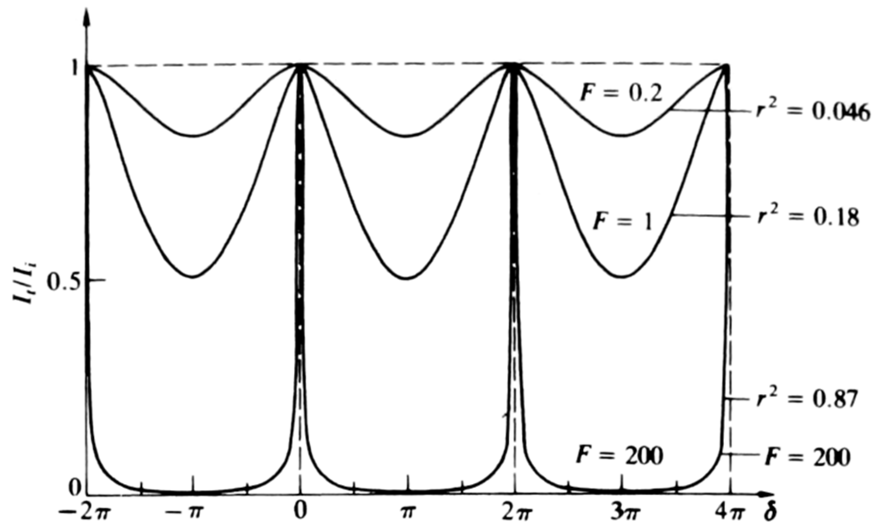
$$\left[1 - \frac{A}{1-R}\right]^2 = 0.0278$$

$$F = 1044$$

Multiple Beam Interference

- How sharp are the peaks?





Width of one line:

$$\gamma = 4/\sqrt{F}$$

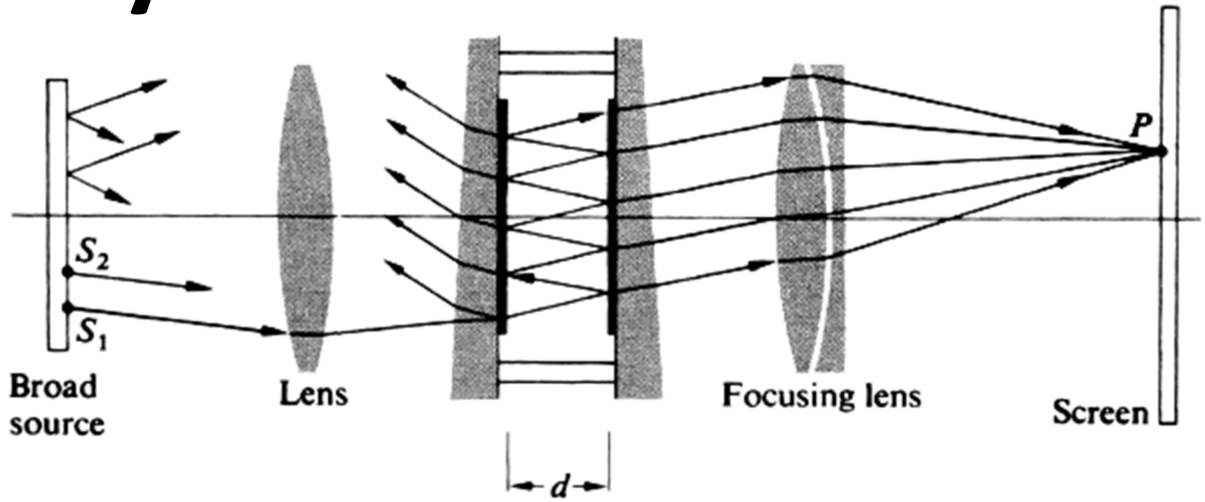
Ratio of line spacing to the width:

$$\mathcal{F} = \frac{2\pi}{\gamma} = \frac{\pi\sqrt{F}}{2}$$

"Finesse" $\mathcal{F}$, not to be confused with the "coefficient of finesse" F.

Previous example: $\mathcal{F} \approx 50$

Fabry-Perot Interferometer



- Phase difference:

$$\delta = \frac{4\pi n_f}{\lambda_0} d\, cos\, \theta_i = 2\pi m$$

$$m\lambda_0 = 2n_f d \cos \theta_i$$

- Differentiate: $m\, \Delta\lambda_0 + \Delta m\, \lambda_0 = 0$

$$\frac{\Delta m}{m} = -\frac{\Delta\lambda_0}{\lambda}$$

Fabry-Perot Interferometer

$$\frac{\lambda_0}{\Delta\lambda_0} = \frac{2\pi m}{\Delta\delta}$$

- Smallest resolvable wavelength difference:

$$(\Delta\lambda_0)_{min} = \frac{\lambda_0(\Delta\delta)_{min}}{2\pi m}$$

- Minimum resolvable phase shift:

$$(\Delta\delta)_{min} \sim \gamma = 4/\sqrt{F}$$

- Chromatic resolving power:

$$\mathcal{R} = \frac{\lambda_0}{(\Delta\lambda_0)_{min}} \approx \mathcal{F}m \approx \mathcal{F}\ \frac{2n_f d}{\lambda_0}$$

Fabry-Perot Interferometer

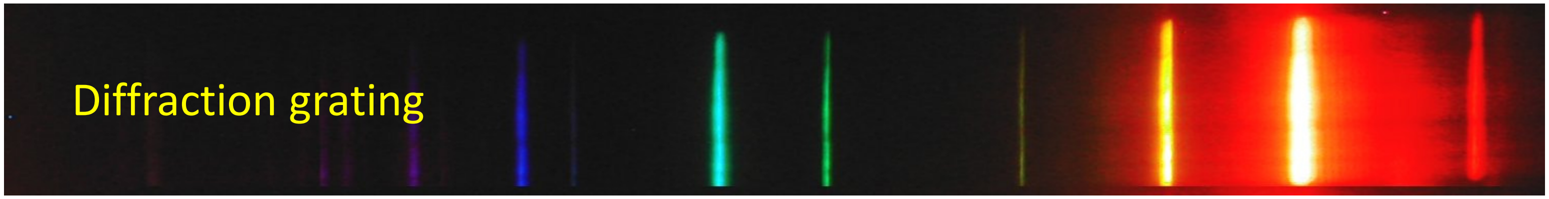
- Typical values:

$$\mathcal{F} = 50$$

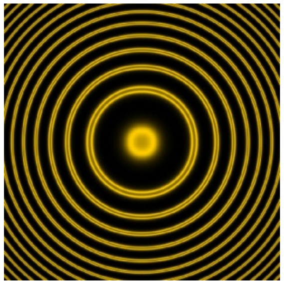
$$n_f d = 1\ cm$$

$$\lambda_0 = 500\ nm$$

$$\mathcal{R} = \frac{2 \times 50 \times 1\ cm}{500\ nm} = 2 \times 10^6$$



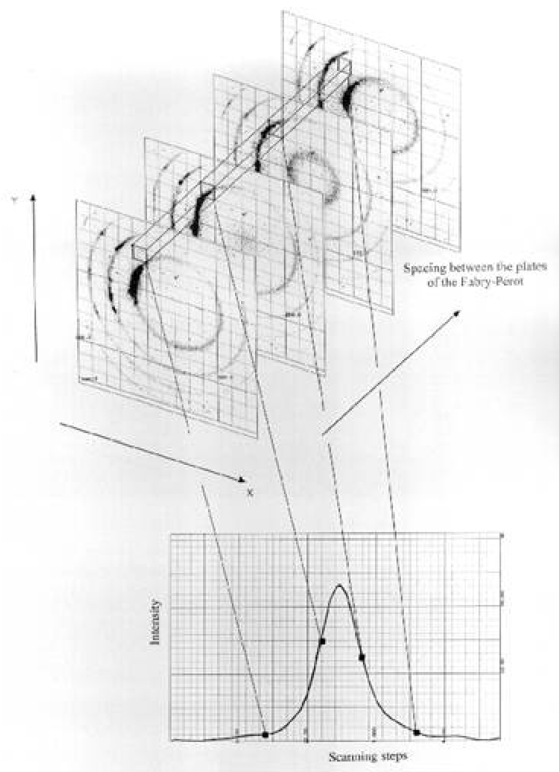
Michelson interferometer

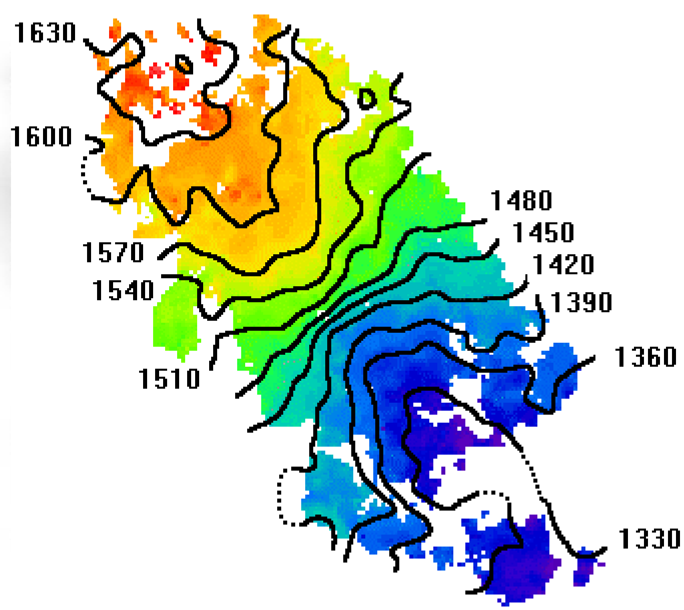


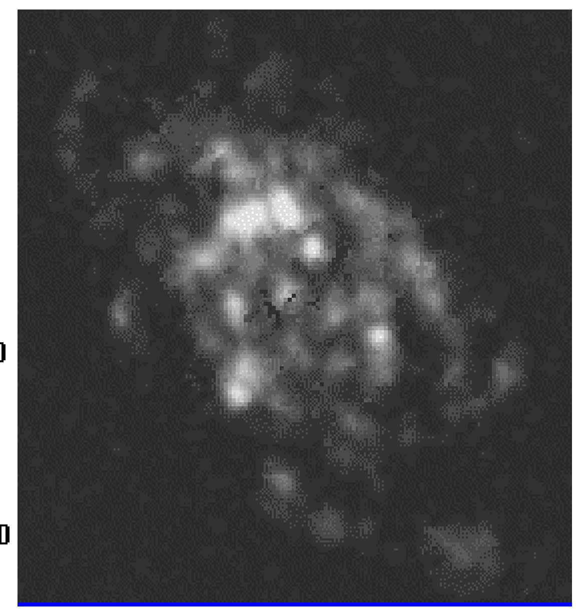
Fabry-Perot interferometer

Fabry-Perot Interferometer

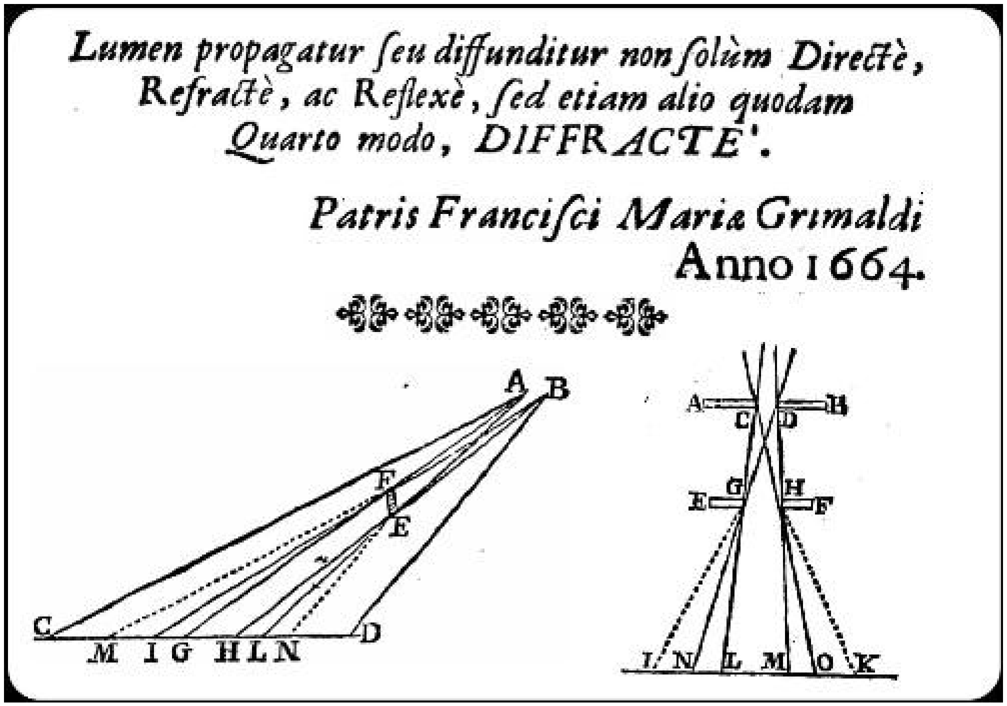
- The effective gap between the surfaces can be adjusted by changing the pressure of a gas, or by means of piezoelectric actuators



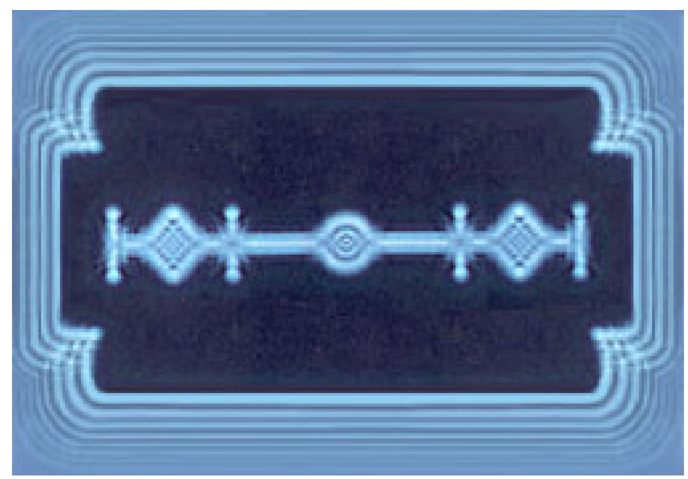




Diffraction


Lumen propagatur ſeu diffunditur non ſolùm Directè, Refractè, ac Reflexè, ſed etiam alio quodam Quarto modo, DIFFRACTE'.

Patris Franciſci Mariæ Grimaldi
Anno 1664.



"Light transmitted or diffused, not only directly, refracted, and reflected, but also in some other way in the fourth, *breaking*."

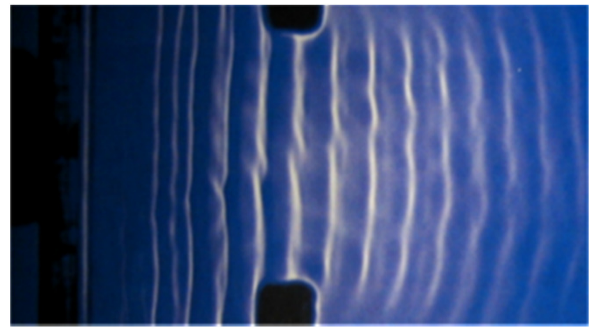
Huygens-Fresnel Principle

- Huygens:
  - Every point on a wave front acts as a point source of secondary spherical waves that have the same phase as the original wave at that point.
- Fresnel:
  - The amplitude of the optical field at any point in the direction of propagation is the superposition of all wavelets, considering their amplitudes and relative phases.

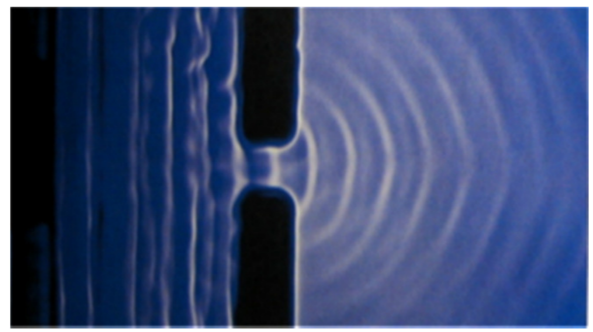
Single Slit Diffraction

Examples with water waves

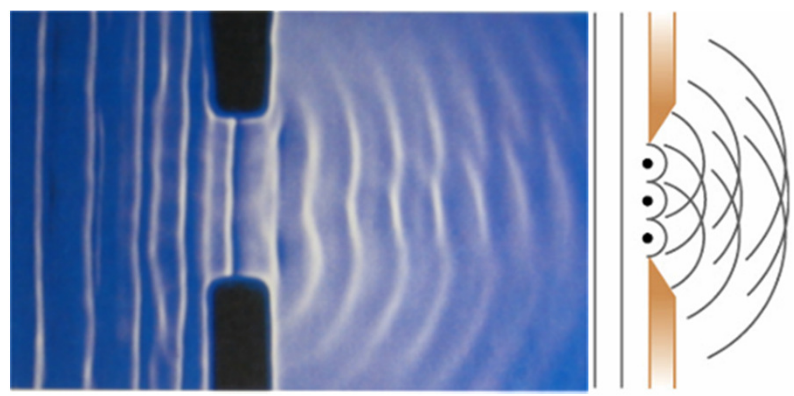
- Wide slit: waves are unaffected



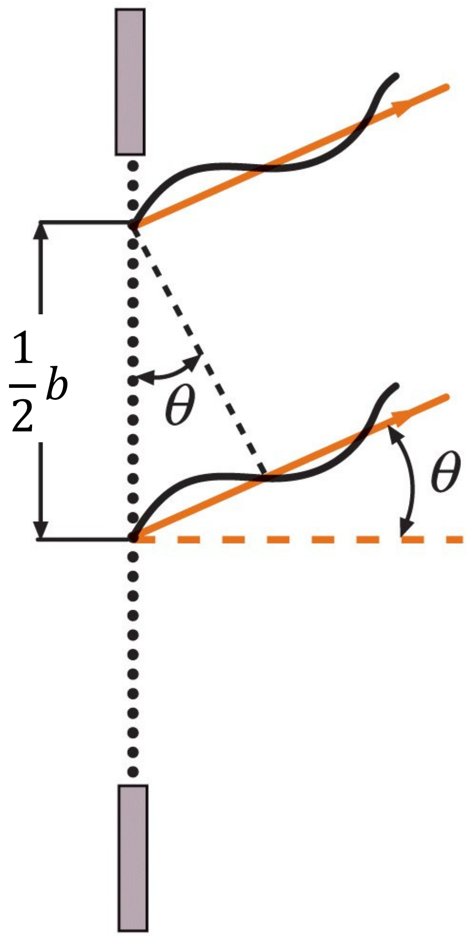
- Narrow slit: source of spherical waves



- In between: multiple interfering point sources



Single Slit Diffraction



Think of the slit as a number of point sources with equal amplitude. Divide the slit into two pieces and think of the interference between light in the upper half and light in the lower half.

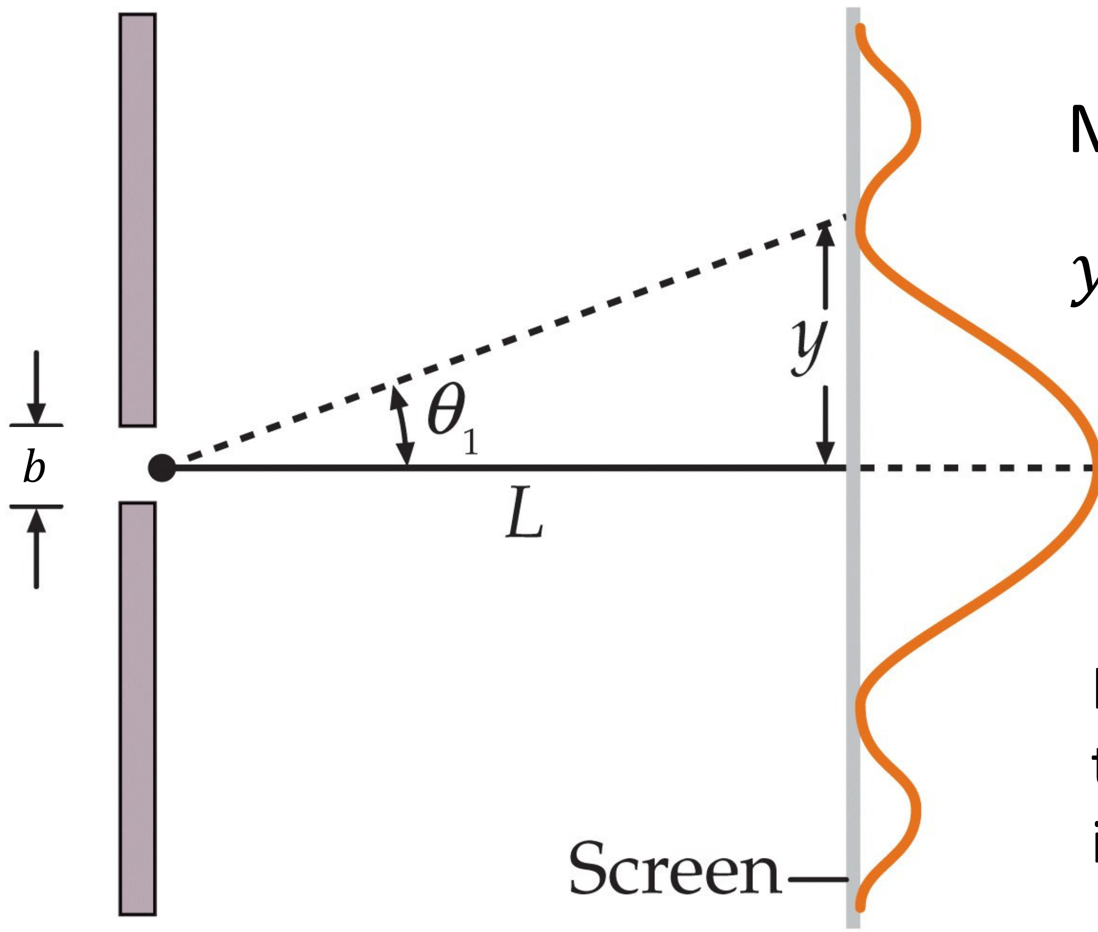
Destructive interference when

$$\frac{b}{2}\sin\theta = \frac{\lambda}{2}$$

Minima when

$$\sin\theta = {}^{\lambda}/_{b}$$

Single Slit Diffraction



$$\sin\theta \approx \tan\theta = y/L$$

Minima located at

$$y = \frac{mL\lambda}{b}, m = 1,2,3,\ldots$$

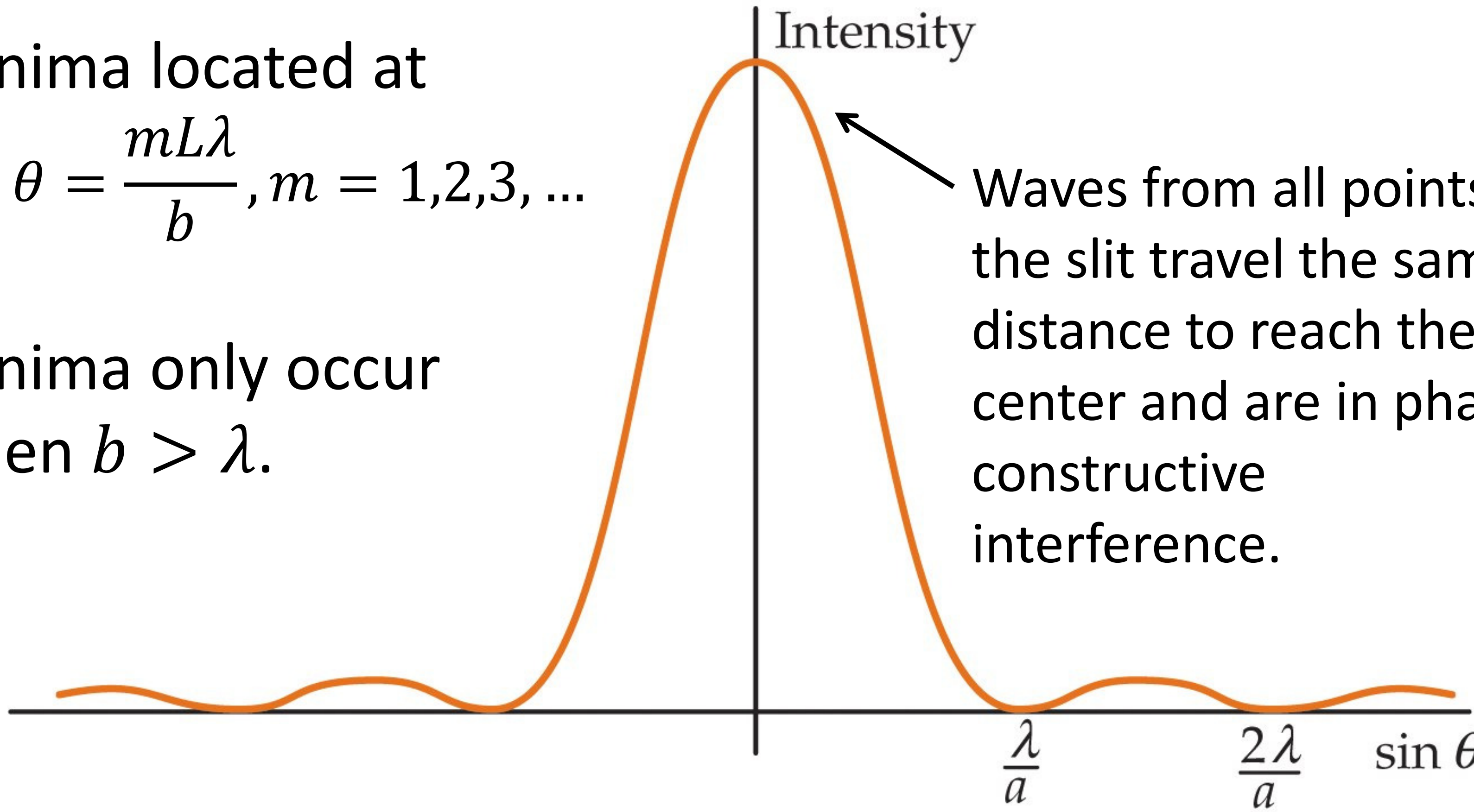
In general, the “width” of the image on the screen is not even close to a.

Single Slit Diffraction

Minima located at

$$\sin\theta = \frac{mL\lambda}{b}, m = 1,2,3, \ldots$$

Minima only occur when $b > \lambda$.

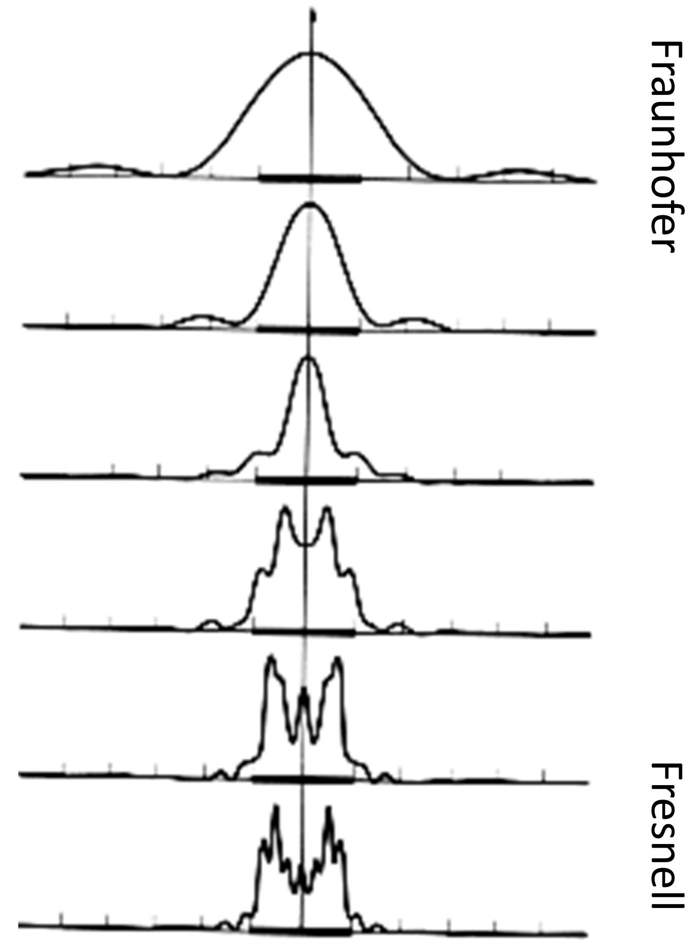


Waves from all points in the slit travel the same distance to reach the center and are in phase: constructive interference.

Fresnel and Fraunhofer Diffraction

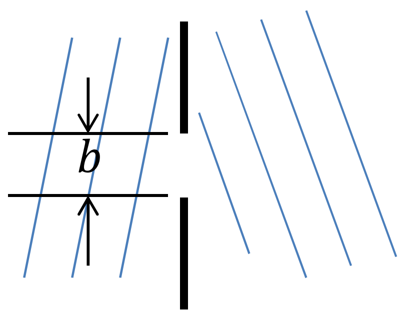
Assumptions about the wave front that impinges on the slit:

- When it's a plane, the phase varies linearly across the slit: Fraunhofer diffraction
- When the phase of the wave front has significant curvature: Fresnel diffraction



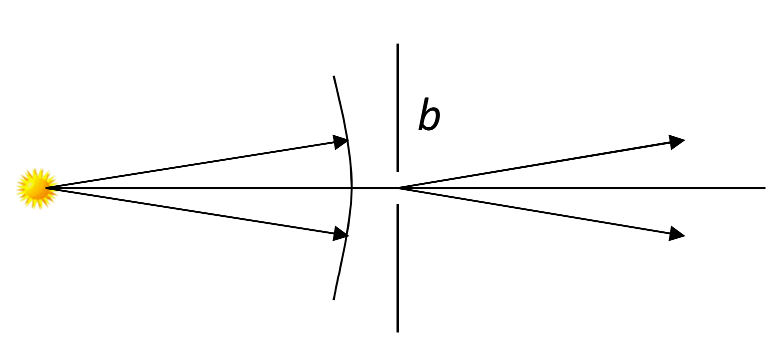
Fresnel and Fraunhofer Diffraction

- Fraunhofer diffraction
  - Far field: $R \gg b^2/\lambda$



  - R is the smaller of the distance to the source or to the screen

- Fresnel Diffraction:
  - Near field: wave front is not a plane at the aperture



Single-Slit Fraunhofer Diffraction

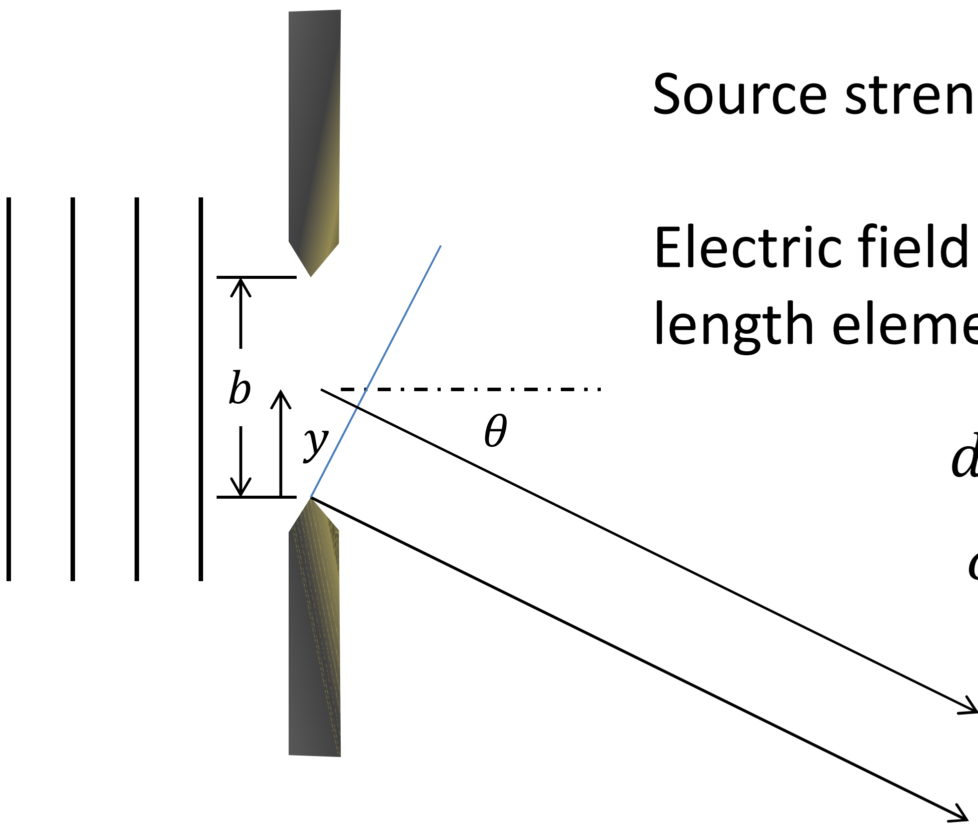
Light with intensity I_0 impinges on a slit with width b

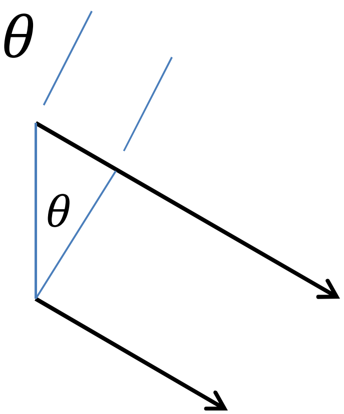
Source strength per unit length: $\mathcal{E}_L$

Electric field at a distance R due to the length element dy:

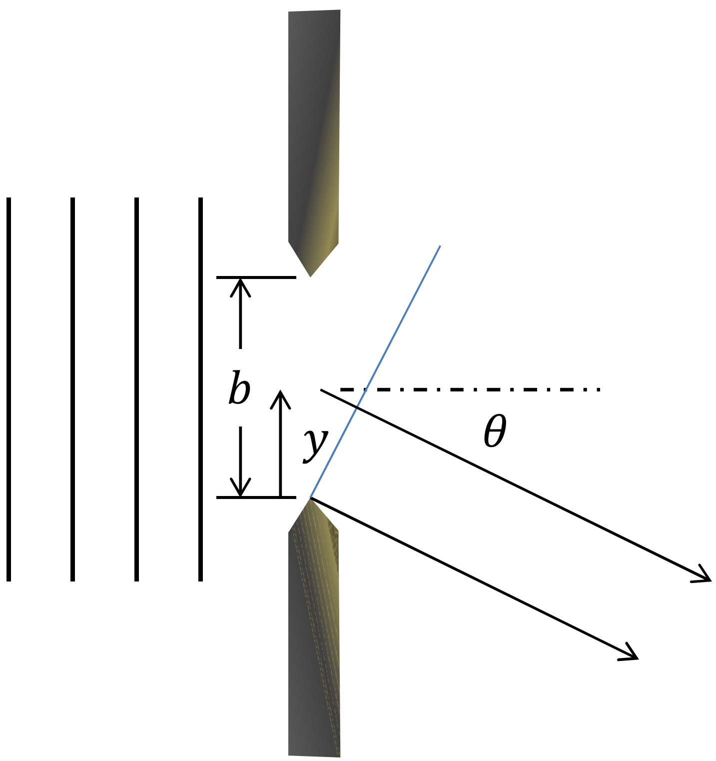
$$dE = \frac{\mathcal{E}_L e^{i\delta(y)} dy}{R}$$

$$\delta(y) = ky \sin\theta$$





Single-Slit Fraunhofer Diffraction



$$dE = \frac{\mathcal{E}_L e^{iky\sin\theta}\,dy}{R}$$

Let $y = 0$ be at the center of the slit.

Integrate from $-b/2$ to $+b/2$:

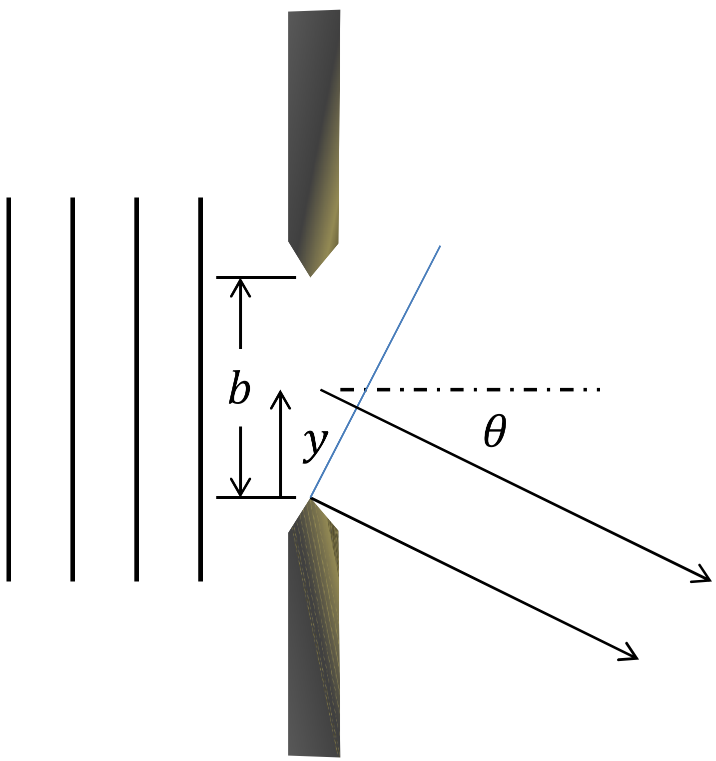
Total electric field:

$$E = \frac{\mathcal{E}_L}{R}\int_{-b/2}^{+b/2} e^{iky\sin\theta}\,dy$$

$$= \frac{\mathcal{E}_L}{R}\frac{e^{i(kb/2)\sin\theta} - e^{-i(kb/2)\sin\theta}}{ik\sin\theta}$$

$$= \frac{\mathcal{E}_L b}{R}\frac{\sin\left(\frac{1}{2}kb\sin\theta\right)}{\frac{1}{2}kb\sin\theta}$$

Single-Slit Fraunhofer Diffraction



$$E = \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta}$$

where

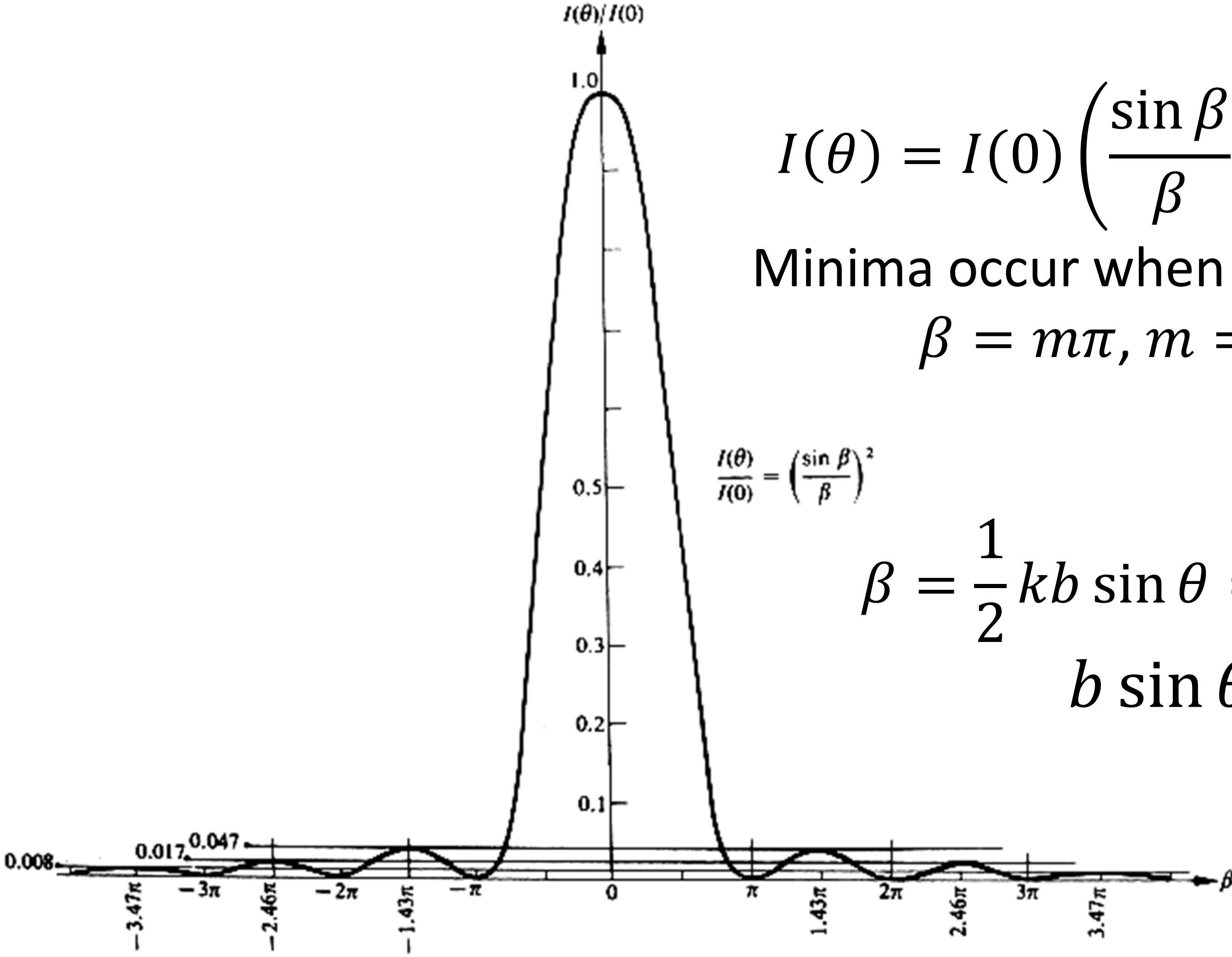
$$\beta = \frac{1}{2} kb \sin\theta$$

The intensity of the light will be

$$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2$$

$$= I(0)\,\mathrm{sinc}^2\,\beta$$

Single-Slit Fraunhofer Diffraction



$$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2 = I(0)\,\mathrm{sinc}^2\,\beta$$

Minima occur when

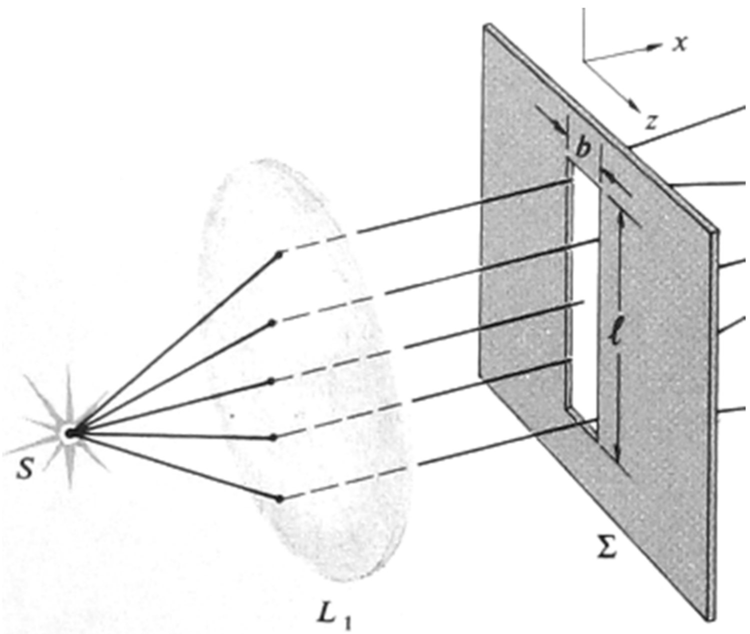
$$\beta = m\pi, m = \pm 1, \pm 2, \cdots$$

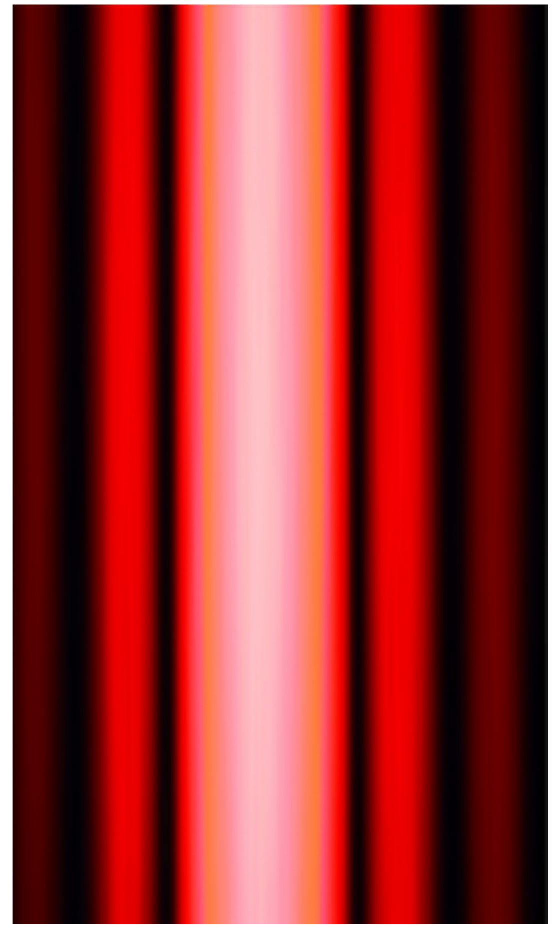
$$\beta = \frac{1}{2}kb\sin\theta = \frac{\pi b}{\lambda}\sin\theta = m\pi$$

$$b\sin\theta = m\lambda$$

Single slit: Fraunhofer diffraction

Adding dimension: long narrow slit
Diffraction most prominent in the narrow direction.





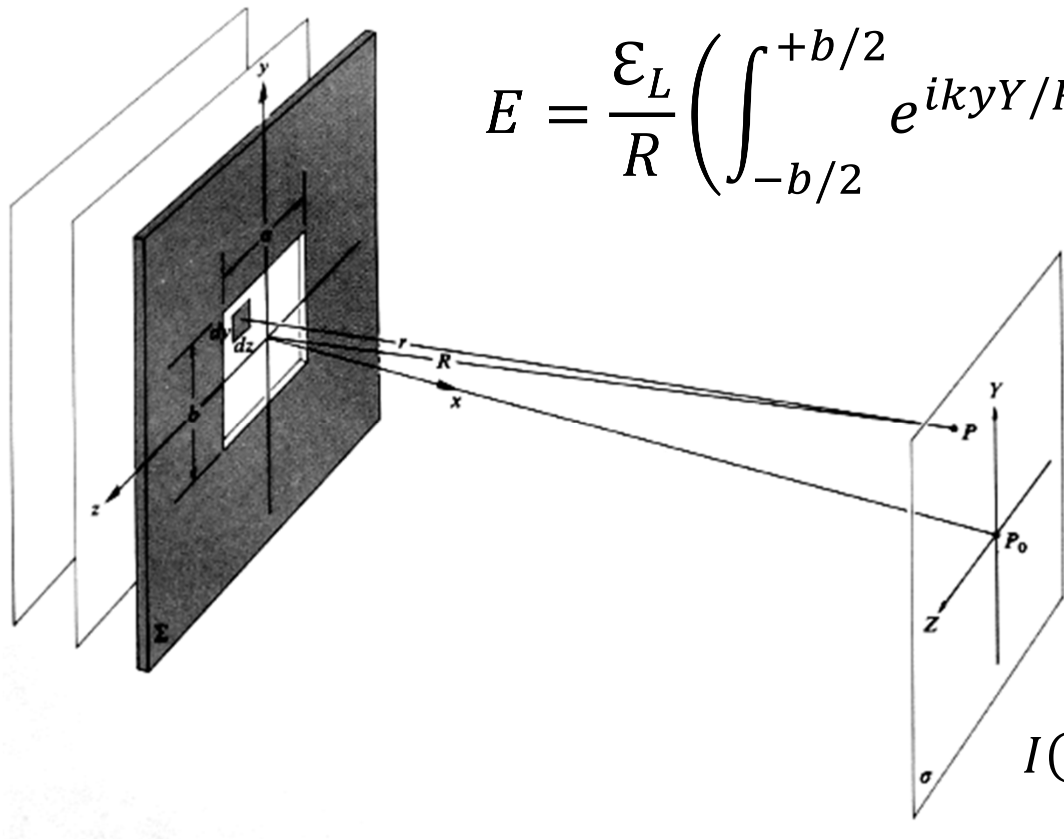
Emerging light has cylindrical symmetry

Rectangular Aperture Fraunhofer Diffraction

Source strength per unit area: $\mathcal{E}_A$

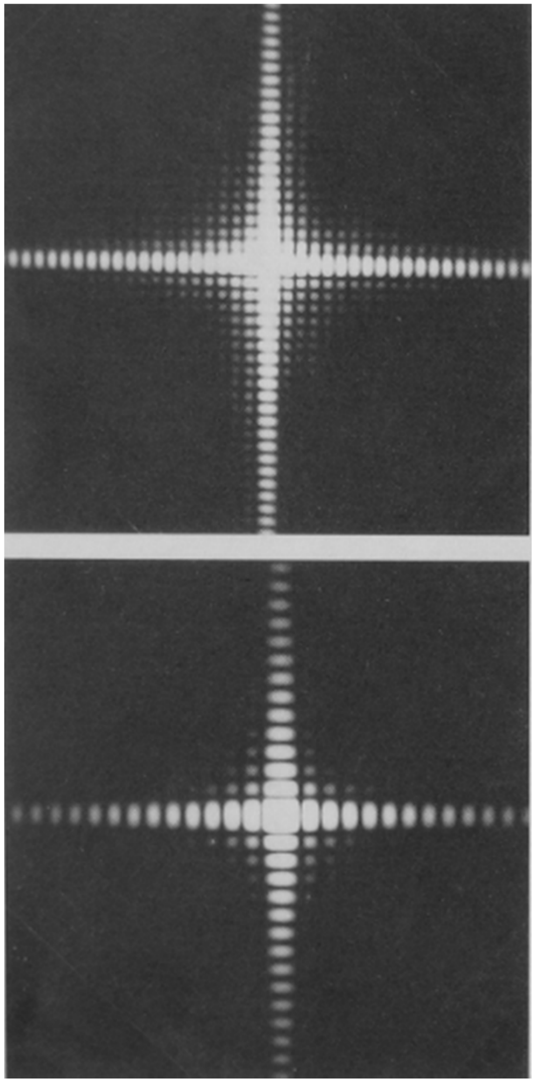
$$dE \approx \frac{\mathcal{E}_L e^{ikyY/R} e^{ikzZ/R} dy dz}{R}$$

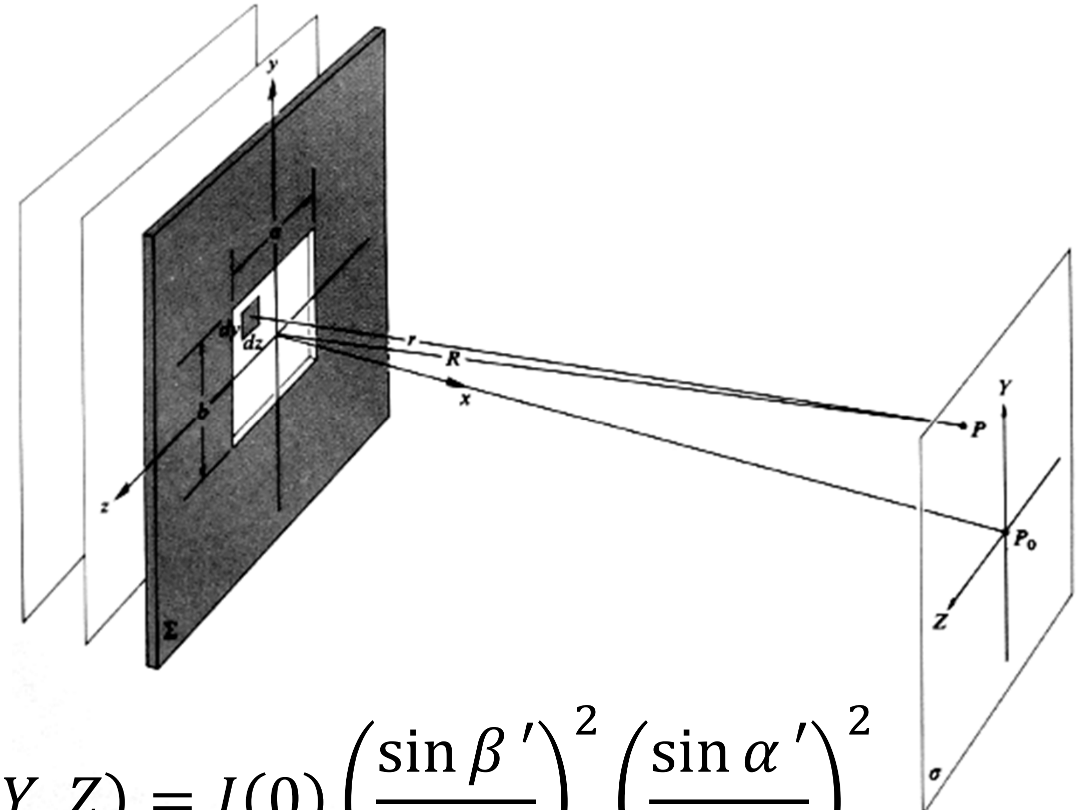
$$E = \frac{\mathcal{E}_L}{R} \left(\int_{-b/2}^{+b/2} e^{ikyY/R} dy \right) \left(\int_{-a/2}^{+a/2} e^{ikzZ/R} dz \right)$$



$$I(Y,Z) = I(0) \left(\frac{\sin \beta'}{\beta'} \right)^2 \left(\frac{\sin \alpha'}{\alpha'} \right)^2$$

Rectangular Aperture





$$I(Y,Z) = I(0)\left(\frac{\sin\beta'}{\beta'}\right)^2\left(\frac{\sin\alpha'}{\alpha'}\right)^2$$

$$\beta' = \frac{1}{2}kbY/R$$

$$\alpha' = \frac{1}{2}kaZ/R$$