PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200
Waves & Oscillations

Lecture 36 – Interference

Spring 2016 Semester

Matthew Jones

Optical Path Length

$$\vec{E}(z,t) = \vec{E}_0 \cos(kz - \omega t)$$

- What is the phase after propagating a distance d?

$$\delta = kd = \frac{2\pi d}{\lambda_0}$$

  - λ_0 is the wavelength in vacuum
  - In a material with index of refraction n, $\lambda = \frac{\lambda_0}{n}$
  - Phase is now

$$\delta' = k'd = \frac{2\pi n d}{\lambda_0}$$

Optical Path Length

- Consider light with wavelength λ_0 in vacuum that is split into two optical paths.
- Phase advance over each path:

$$\delta_1 = \frac{2\pi n_1 d_1}{\lambda_0} \qquad\qquad \delta_2 = \frac{2\pi n_2 d_2}{\lambda_0}$$

- Phase difference between two paths:

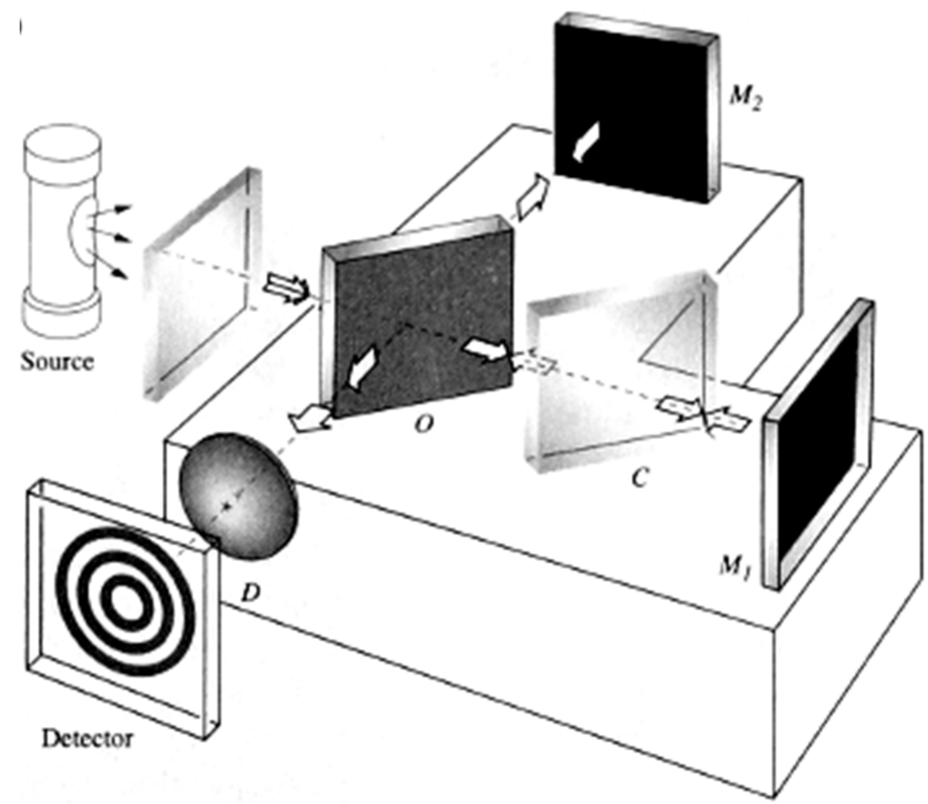
$$\Delta\delta = \frac{2\pi}{\lambda_0}(n_1 d_1 - n_2 d_2)$$

- Constructive interference when

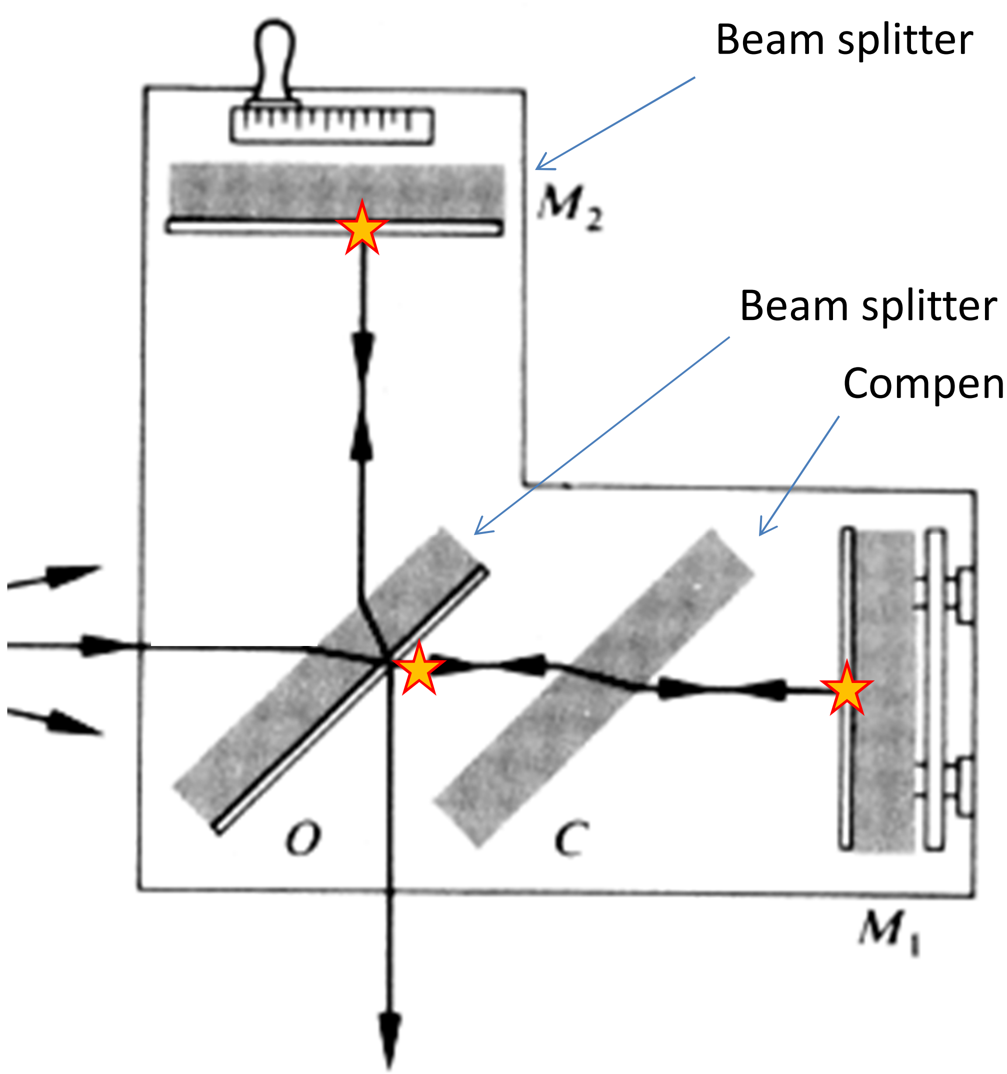
$$n_1 d_1 - n_2 d_2 = m\lambda_0 \text{ where } m = 0,1,2, \ldots$$

- The *optical path length* is defined $\ell = nd$.

Michelson Interferometer



Michelson Interferometer



Phase reversal: ☆

Difference in path length:

$$d_1 - d_2$$

Destructive interference:

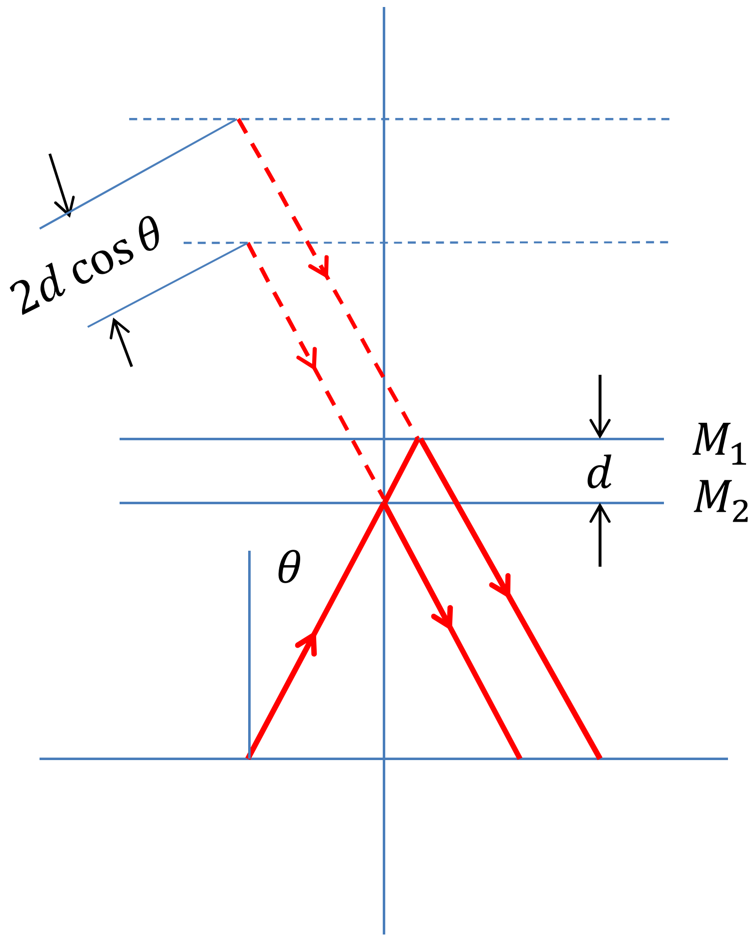
$$d_1 - d_2 = m\lambda$$

Constructive interference:

$$d_1 - d_2 = \left(m + \frac{1}{2}\right)\lambda$$

Michelson Interferometer

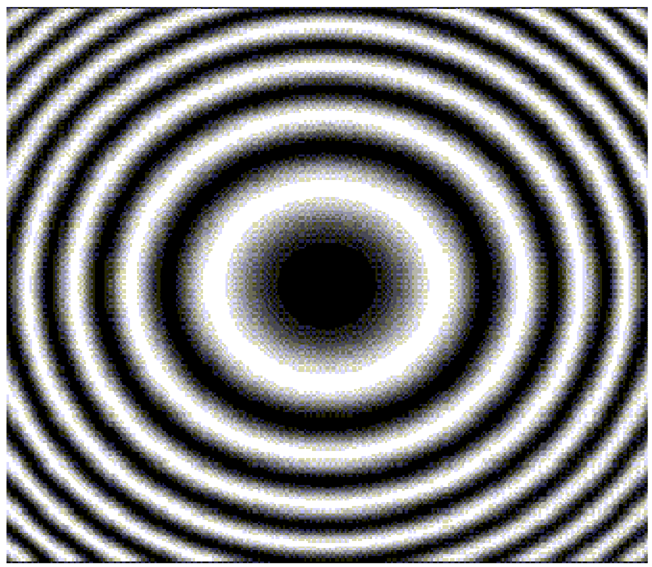
- Equivalent optics:



Michelson Interferometer

- Bright fringes occur when

$$\delta = 2\pi\left(\frac{2d}{\lambda\cos\theta} + \frac{1}{2}\right) = 2\pi m$$



Michelson Interferometer

- How does the position of a fringe change when the path length changes?

$$\frac{2d}{\lambda \cos\theta} = m + \frac{1}{2}$$

$$2d = \lambda \cos\theta \left(m + \frac{1}{2}\right)$$

$$2\Delta d = -\lambda \sin\theta \left(m + \frac{1}{2}\right)\Delta\theta$$

$$\frac{\Delta\theta}{\Delta d} = -\frac{2}{\left(m + \frac{1}{2}\right)\lambda \sin\theta}$$

Michelson Interferometer

- Application: Consider two closely spaced wavelengths, λ and λ'
- Bright fringes from one wavelength occur when

$$\frac{2d}{\lambda} = m$$

- Bright fringes from the other wavelength occur when

$$\frac{2d}{\lambda'} = m'$$

- The two fringes will coincide when

$$\frac{2d}{\lambda} = \frac{2d}{\lambda'} + N$$

Michelson Interferometer

- Adjust the position of the movable mirror so that the next set of fringes coincide

$$\frac{2d'}{\lambda} = \frac{2d'}{\lambda'} + N + 1$$

- Subtract these:

$$\frac{2d'}{\lambda} - \frac{2d}{\lambda} = \frac{2d'}{\lambda'} - \frac{2d}{\lambda'} + 1$$

$$\lambda' - \lambda = \frac{\lambda\lambda'}{2\Delta d} \approx \frac{\lambda^2}{2\Delta d}$$

- For the yellow sodium line,

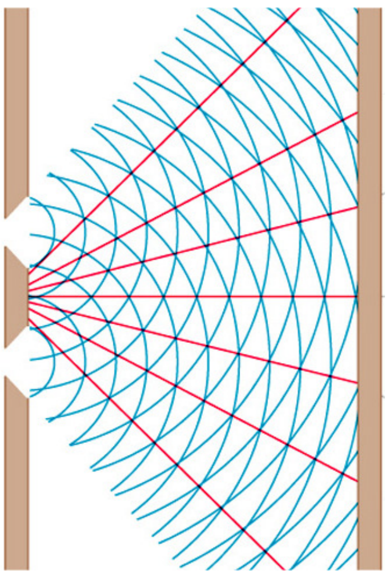
$$\left.\begin{aligned} \lambda &= 588.991\,nm \\ \lambda' &= 589.595\,nm \end{aligned}\right\} \Delta\lambda = 0.604\,nm$$

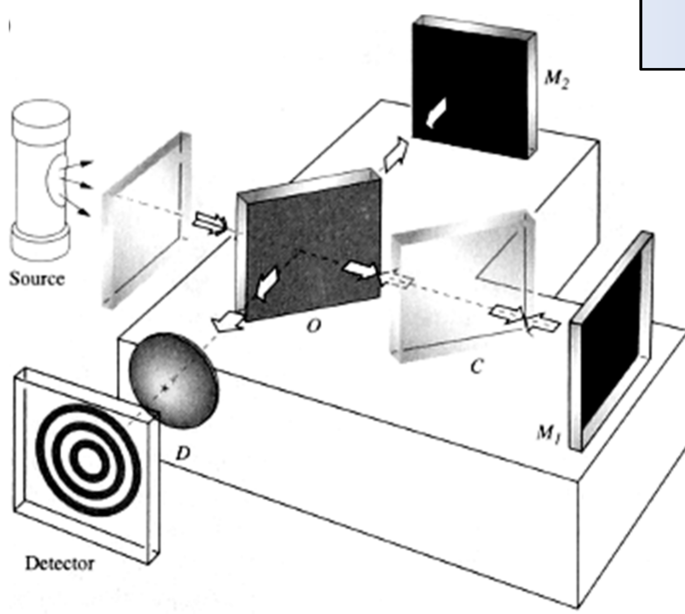
$$\Delta d = \lambda^2/2\Delta\lambda = 287{,}472\,nm = 0.287\,mm$$

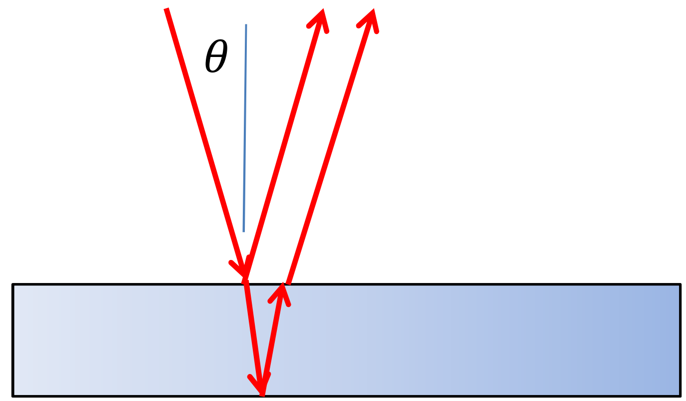
Michelson Interferometer

Multiple Beam Interference

- Previously we considered only two interfering beams:



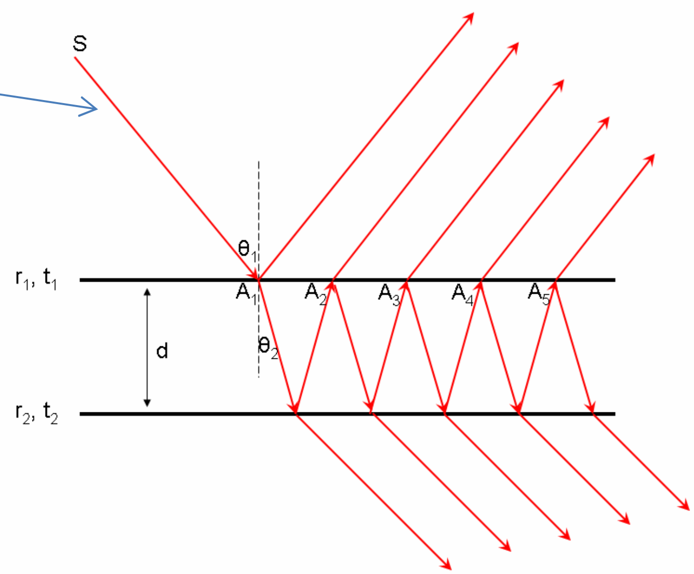




Multiple Beam Interference

- In many situations, a coherent beam can interfere with itself multiple times
- Consider a beam incident on a thin film
  - Some component of the light will be reflected at each surface and some will be transmitted

Incident light

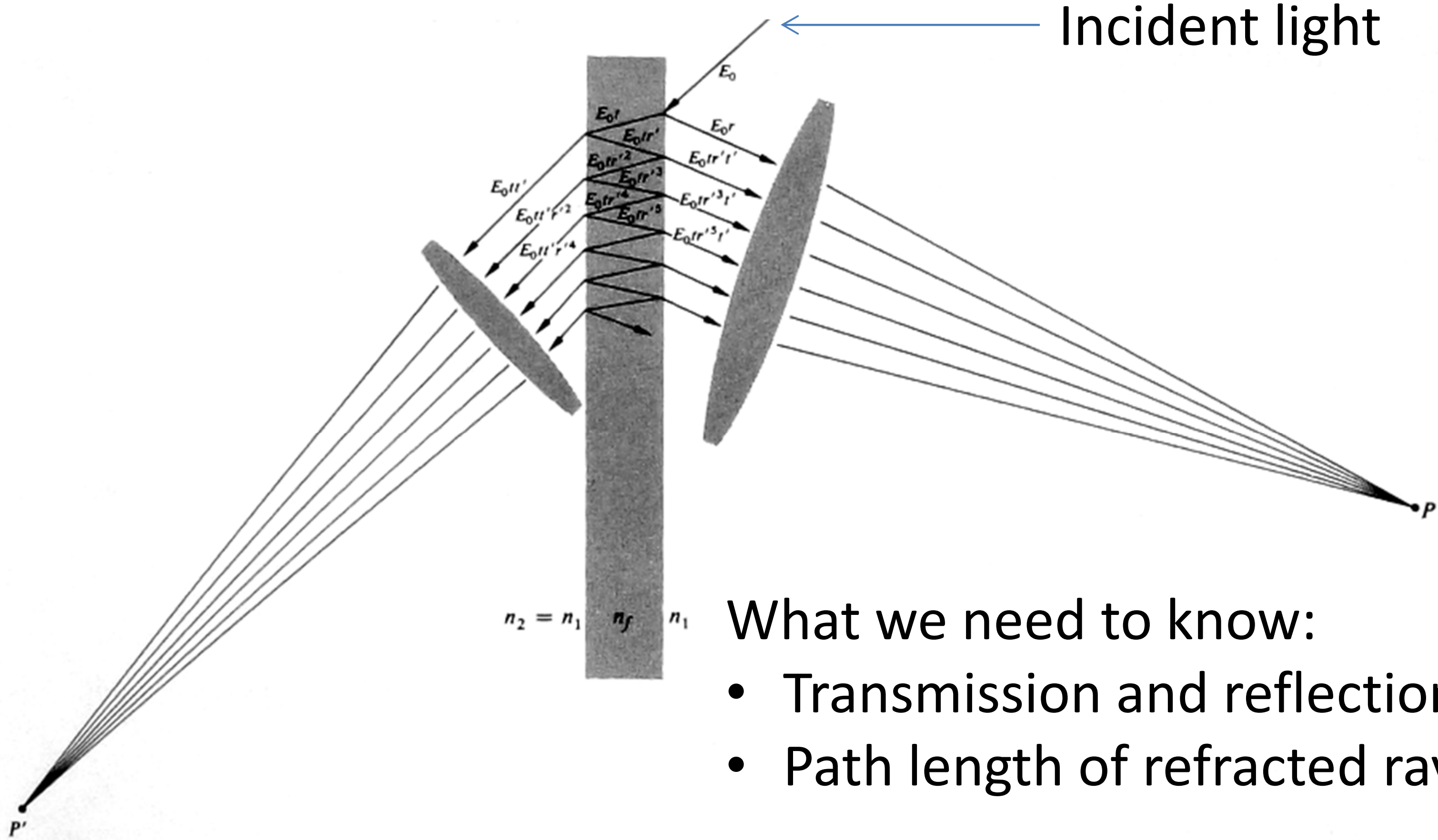


Each transmitted beam will have a different phase relative to the adjacent beams.

What is the total intensity of the reflected light?

Multiple Beam Interference

- All transmitted and reflected rays will be parallel
- They can be focused onto points P and P’ by lenses:

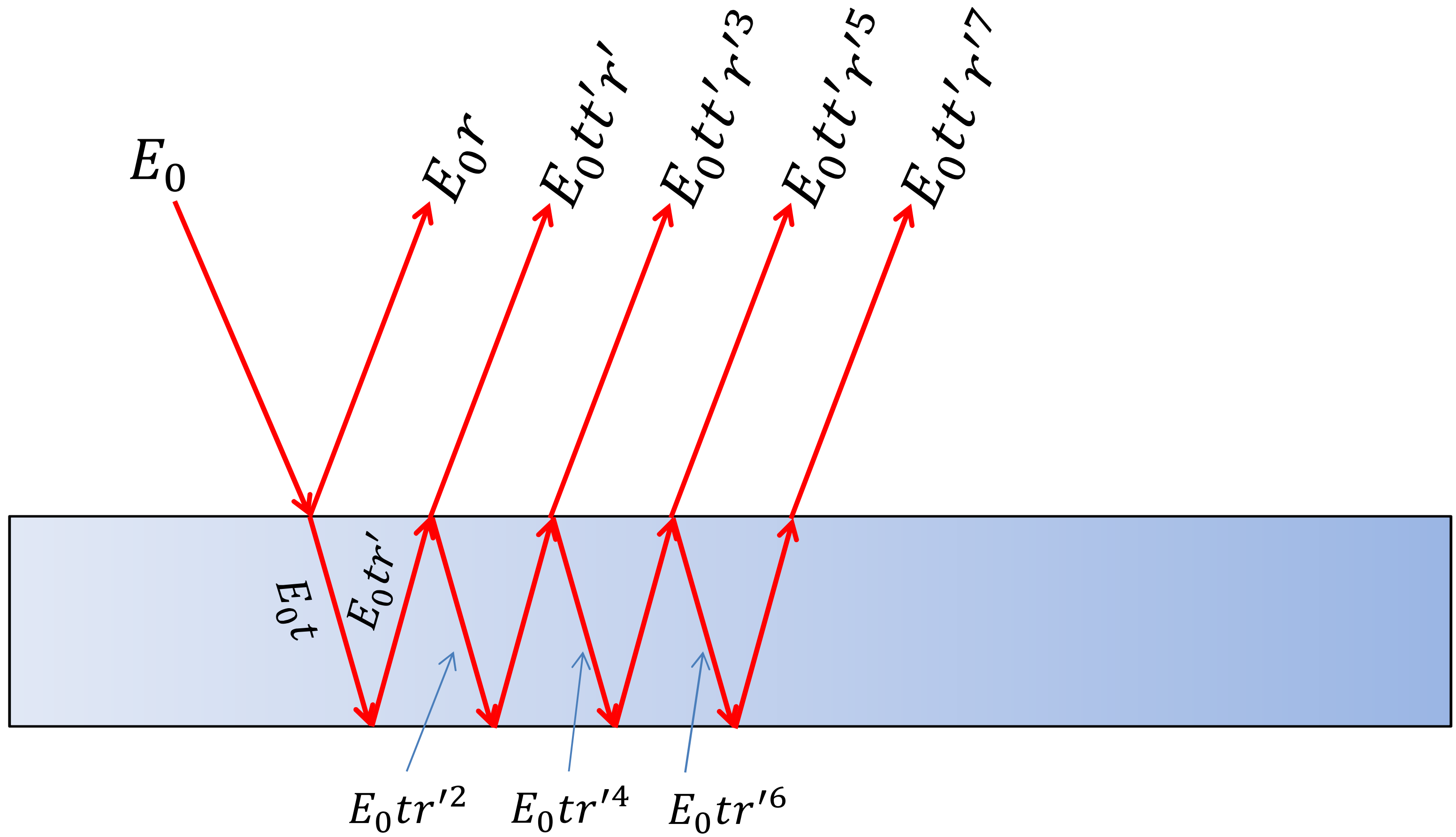


What we need to know:

- Transmission and reflection coefficients
- Path length of refracted rays in the film

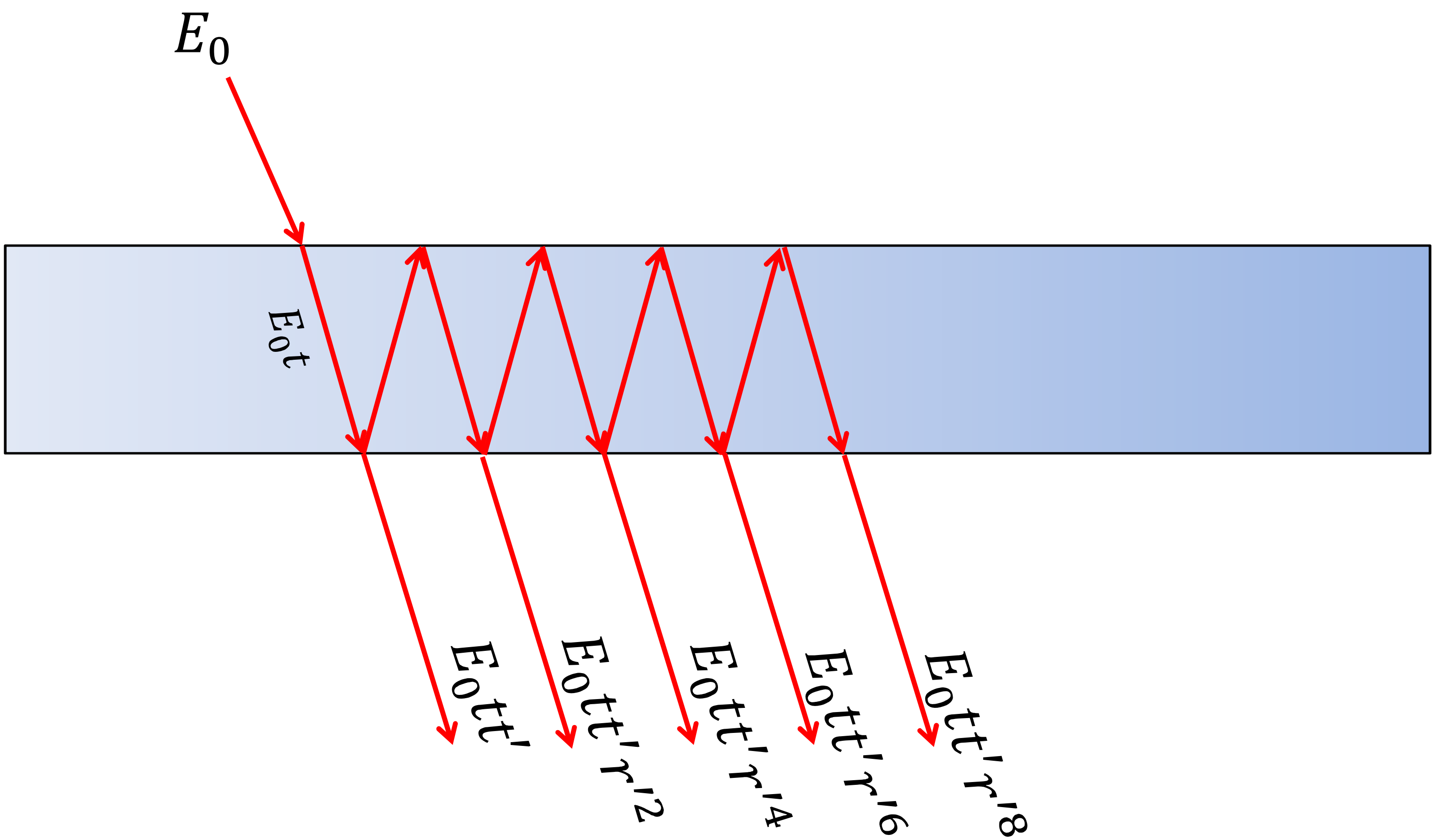
Multiple Beam Interference

- Reflection coefficients: r and r'
- Transmission coefficients: t and t'



Multiple Beam Interference

- Reflection coefficients: r and r'
- Transmission coefficients: t and t'



Multiple Beam Interference

- The additional phase in the film is always the same:

$$\delta = \frac{2nkd}{\cos\theta_t}$$

- If the initial phase is zero, then

$$E_{1r} = E_0 r e^{i\omega t}$$
$$E_{2r} = E_0 t t' r' e^{i(\omega t - \delta)}$$
$$E_{3r} = E_0 t t' r'^3 e^{i(\omega t - 2\delta)}$$
$$E_{4r} = E_0 t t' r'^5 e^{i(\omega t - 3\delta)}$$
$$\cdots$$

- In general:

$$E_{Nr} = E_0 e^{i\omega t} t t' \, r'^{2N-3} e^{-i(N-1)\delta}$$
$$= E_0 e^{-i\omega t} t t' r' e^{-i\delta} \left(r'^2 e^{-i\delta}\right)^{N-2}$$

Multiple Beam Interference

- The total electric field on one side of the film:

$$E_r = E_0 e^{i\omega t} r + E_0 e^{i\omega t} tt'r'e^{-i\delta} \times$$
$$\left[1 + r'^2 e^{-i\delta} + \left(r'^2 e^{-i\delta}\right)^2 + \left(r'^2 e^{-i\delta}\right)^3 + \cdots\right]$$

- This is in infinite sum of the form:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad (\text{when } |z| < 1)$$

- Total electric field:

$$E_r = E_0 e^{i\omega t}\left[r + \frac{tt'r'e^{-i\delta}}{1 - r'^2 e^{-i\delta}}\right]$$

Multiple Beam Interferometry

- Simplifications:

$$r' = -r$$
$$tt' = 1 - r^2$$

- Total electric field:

$$E_r = E_0 e^{i\omega t} r \left[1 - \frac{(1 - r^2) e^{-i\delta}}{1 - r^2 e^{-i\delta}} \right]$$
$$= E_0 e^{i\omega t} r \left[\frac{1 - r^2 e^{-i\delta} - e^{-i\delta} + r^2 e^{-i\delta}}{1 - r^2 e^{-i\delta}} \right]$$
$$= E_0 e^{i\omega t} r \left[\frac{1 - e^{-i\delta}}{1 - r^2 e^{-i\delta}} \right]$$

Multiple Beam Interferometry

- The intensity of the light is $I_r \propto |E_r|^2$

$$I_r = I_0 \left\{ r \left[\frac{1 - e^{-i\delta}}{1 - r^2 e^{-i\delta}} \right] \right\}^* \left\{ r \left[\frac{1 - e^{-i\delta}}{1 - r^2 e^{-i\delta}} \right] \right\}$$

$$= I_0 r^2 \frac{(1 - e^{i\delta})(1 - e^{-i\delta})}{(1 - r^2 e^{i\delta})(1 - r^2 e^{-i\delta})}$$

$$= I_0 \frac{2r^2(1 - \cos\delta)}{(1 + r^4) - 2r^2 \cos\delta}$$

- The intensity of the transmitted light is $I_t \propto |E_t|^2$

$$I_t = I_0 \frac{1 - r^2}{(1 + r^4) - 2r^2 \cos\delta}$$

Multiple Beam Interferometry

- One more identity will clean this up a bit:

$$\cos\delta = 1 - 2\sin^2(\delta/2)$$

- Reflected intensity:

$$I_r = I_0\,\frac{F\sin^2(\delta/2)}{1 + F\sin^2(\delta/2)}$$

- Transmitted intensity:

$$I_t = I_0\,\frac{1}{1 + F\sin^2(\delta/2)}$$

- The parameter $F = \left(\frac{2r}{1-r^2}\right)^2$ is called the ***coefficient of finesse***
- Notice that $I_0 = I_r + I_t$
  - We assumed that no energy was lost in the film

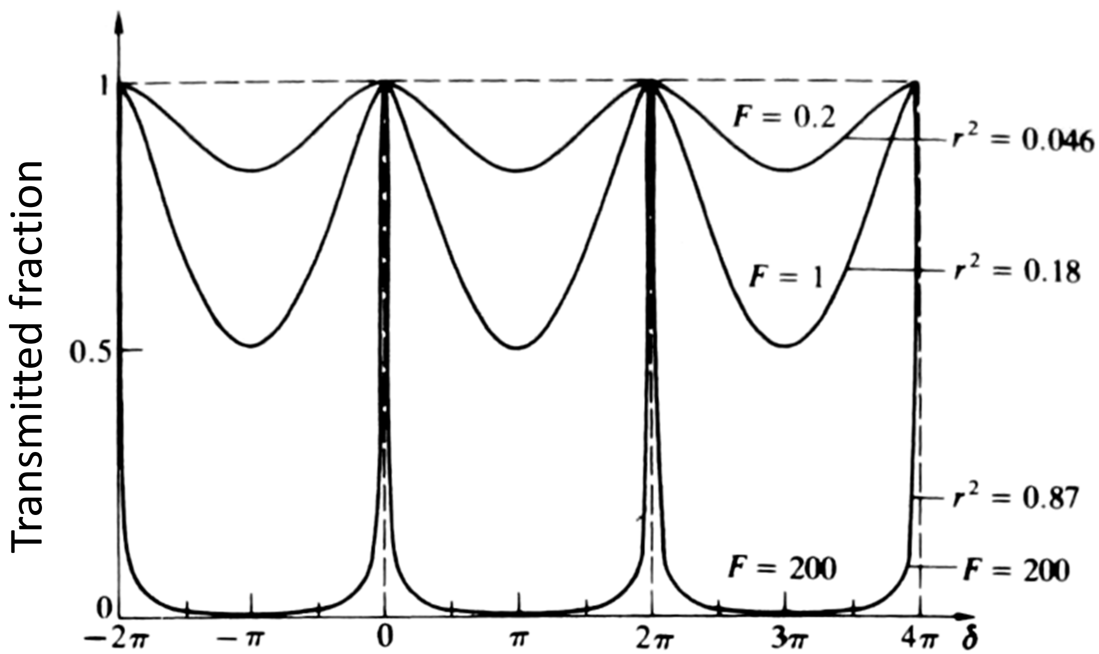
Multiple Beam Interferometry

The function

$$\mathcal{A}(\theta) = \frac{1}{1 + F\sin^2(\delta/2)}$$

is called the Airy function.

Remember that δ is a function of the angle of incidence, θ.



Multiple Beam Interferometry

- In practice, some fraction of the light will be absorbed
- Absorptance, A, is defined by:

$$T + R + A = 1$$

- This modifies the transmitted intensity:

$$I_t = I_0 \left[1 - \frac{A}{1-R}\right]^2 \mathcal{A}(\theta)$$

- Example: silver film, 50 nm thick, deposited on glass

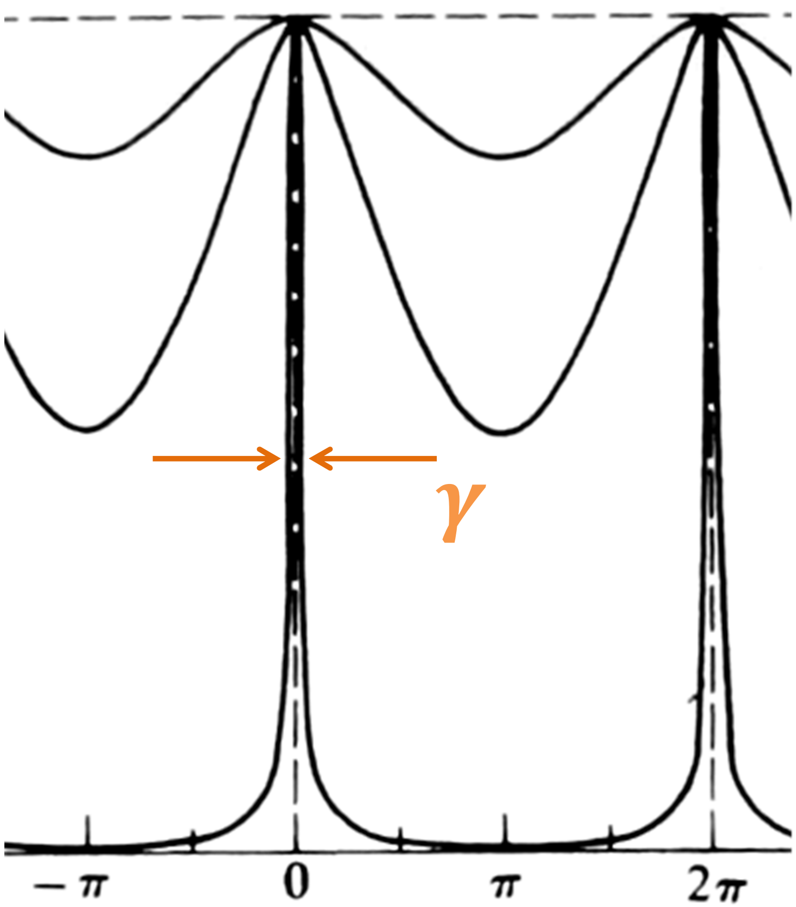
$$R = 0.94, T = 0.01, A = 0.05$$

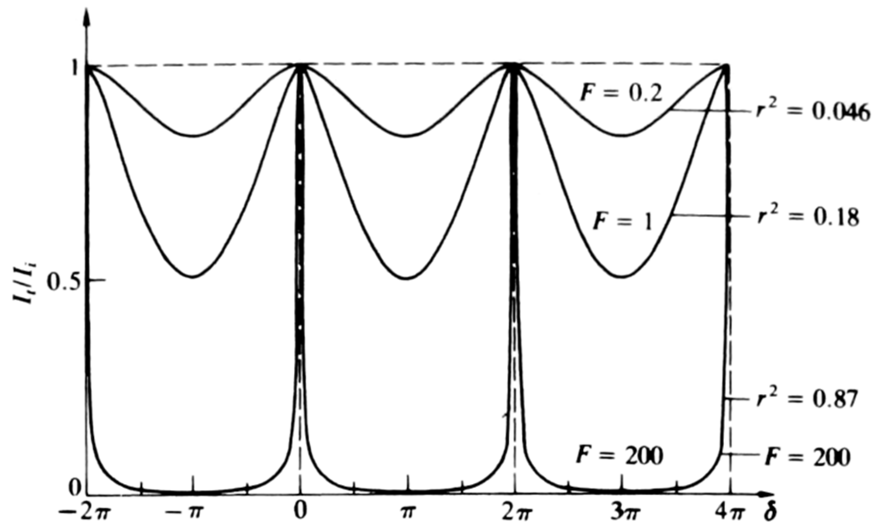
$$\left[1 - \frac{A}{1-R}\right]^2 = 0.0278$$

$$F = 1044$$

Multiple Beam Interference

- How sharp are the peaks?





Width of one line:

$$\gamma = 4/\sqrt{F}$$

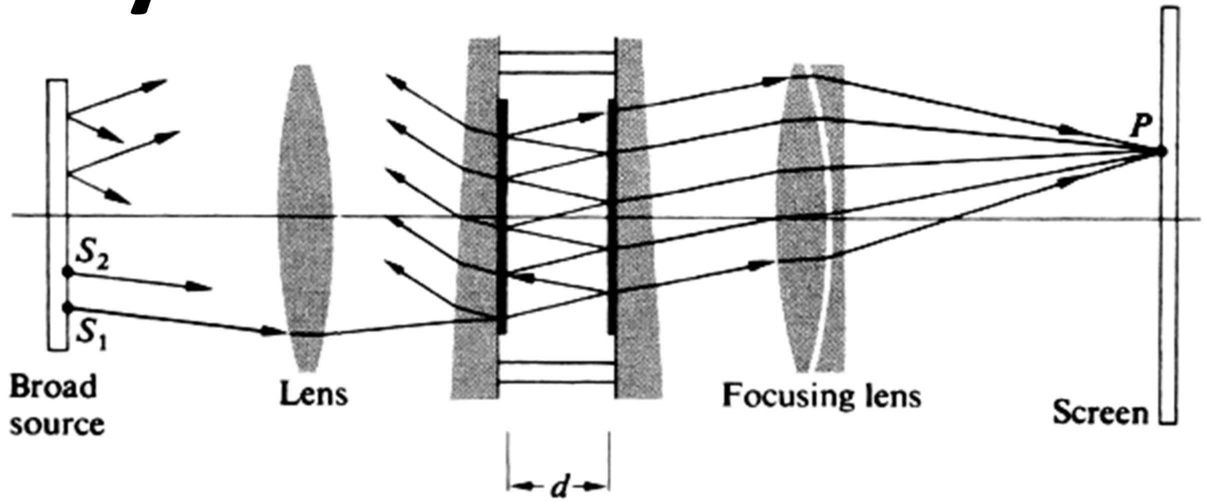
Ratio of line spacing to the width:

$$\mathcal{F} = \frac{2\pi}{\gamma} = \frac{\pi\sqrt{F}}{2}$$

"Finesse" $\mathcal{F}$, not to be confused with the "coefficient of finesse" F.

Previous example: $\mathcal{F} \approx 50$

Fabry-Perot Interferometer



- Phase difference:

$$\delta = \frac{4\pi n_f}{\lambda_0} d\, cos\, \theta_i = 2\pi m$$

$$m\lambda_0 = 2 n_f d \cos\theta_i$$

- Differentiate: $m\, \Delta\lambda_0 + \Delta m\, \lambda_0 = 0$

$$\frac{\Delta m}{m} = -\frac{\Delta\lambda_0}{\lambda}$$

Fabry-Perot Interferometer

$$\frac{\lambda_0}{\Delta\lambda_0} = \frac{2\pi m}{\Delta\delta}$$

- Smallest resolvable wavelength difference:

$$(\Delta\lambda_0)_{min} = \frac{\lambda_0(\Delta\delta)_{min}}{2\pi m}$$

- Minimum resolvable phase shift:

$$(\Delta\delta)_{min} \sim \gamma = 4/\sqrt{F}$$

- Chromatic resolving power:

$$\mathcal{R} = \frac{\lambda_0}{(\Delta\lambda_0)_{min}} \approx \mathcal{F}m \approx \mathcal{F}\ \frac{2n_f d}{\lambda_0}$$

Fabry-Perot Interferometer

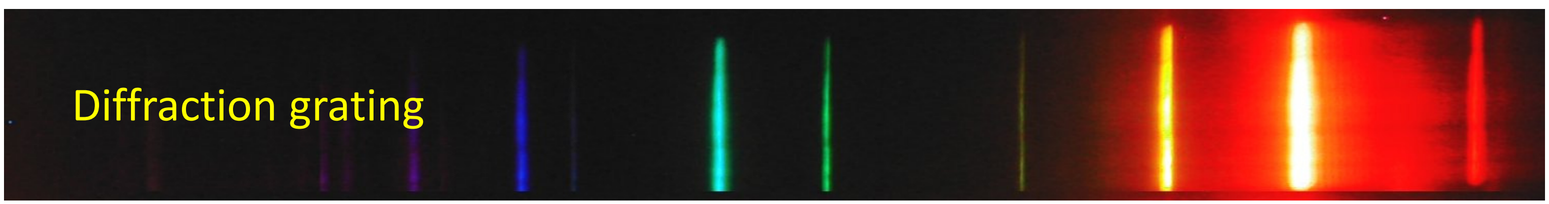
- Typical values:

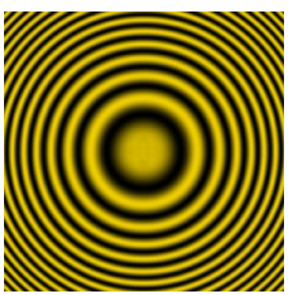
$$\mathcal{F} = 50$$

$$n_f d = 1\ cm$$

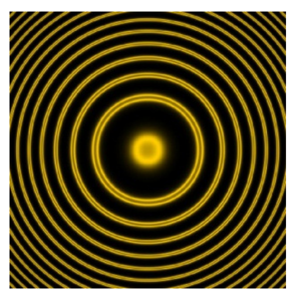
$$\lambda_0 = 500\ nm$$

$$\mathcal{R} = \frac{2 \times 50 \times 1\ cm}{500\ nm} = 2 \times 10^6$$





Michelson interferometer



Fabry-Perot interferometer

Fabry-Perot Interferometer

- The effective gap between the surfaces can be adjusted by changing the pressure of a gas, or by means of piezoelectric actuators