PURDUE DEPARTMENT OF PHYSICS

Physics 42200
Waves & Oscillations

Lecture 32 – Electromagnetic Waves

Spring 2016 Semester

Matthew Jones

Electromagnetism

- Geometric optics overlooks the wave nature of light.
  - Light inconsistent with longitudinal waves in an ethereal medium
  - Still an excellent approximation when feature sizes are large compared with the wavelength of light
- But geometric optics could not explain
  - Polarization
  - Diffraction
  - Interference
- A unified picture was provided by Maxwell c. 1864

Maxwell's Equations (1864)

$$\oint_S \hat{n} \cdot \vec{E}\, dA = \frac{Q_{inside}}{\epsilon_0} \quad \text{①}$$

$$\oint_S \hat{n} \cdot \vec{B}\, dA = 0 \quad \text{②}$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d\phi_m}{dt} \quad \text{③}$$

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\phi_e}{dt} \quad \text{④}$$

Maxwell's Equations in Free Space

In "free space" where there are no electric charges or sources of current, Maxwell's equations are quite symmetric:

$$\oint_S \hat{n} \cdot \vec{E}\, dA = 0$$

$$\oint_S \hat{n} \cdot \vec{B}\, dA = 0$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d\phi_m}{dt}$$

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \varepsilon_0 \frac{d\phi_e}{dt}$$

Velocity of Electromagnetic Waves

$$\frac{\partial^2 y}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

Speed of wave propagation is

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$= \frac{1}{\sqrt{(4\pi \times 10^{-7} N/A^2)(8.854 \times 10^{-12}\, C^2/N \cdot m)}}$$

$$= 2.998 \times 10^8 \text{ m/s}$$

(speed of light)

Electromagnetic Waves

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

- A solution is $E_x(z,t) = E_0 \sin(kz - \omega t)$
  where $\omega = kc = 2\pi c/\lambda$
- What is the magnetic field?

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z} = -kE_0 \cos(kz - \omega t)$$

$$B_y(x,t) = \frac{k}{\omega}E_0 \sin(kx - \omega t)$$
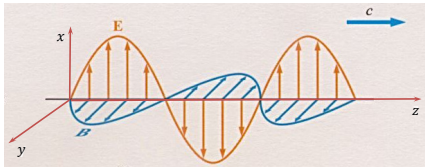
Electromagnetic Waves

- $\vec{E}$, $\vec{B}$ and $\vec{v}$ are mutually perpendicular.
- In general, the direction is
$$\hat{s} = \hat{E} \times \hat{B}$$

Energy in Electromagnetic Waves

Energy density of electric and magnetic fields:
$$u_e = \frac{1}{2}\epsilon_0 E^2 \qquad u_m = \frac{1}{2\mu_0}B^2$$

For an electromagnetic wave, $B = E/c = E\sqrt{\mu_0 \epsilon_0}$
$$u_m = \frac{1}{2\mu_0}B^2 = \frac{1}{2}\epsilon_0 E^2 = u_e$$

The total energy density is
$$u = u_m + u_e = \epsilon_0 E^2$$

Intensity of Electromagnetic Waves

- Intensity is defined as the average power transmitted per unit area.

  Intensity = Energy density × wave velocity

$$I = \epsilon_0 c \langle E^2 \rangle = \frac{\langle E^2 \rangle}{\mu_0 c}$$
$$\mu_0 c = 377 \ \Omega \equiv Z_0$$
(Impedance of free space)

Polarization

- Light is an oscillating electromagnetic field
- The electric field has a direction

$$\vec{E}(x,t) = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t)$$

- No need to specify the magnetic field direction:

$$\vec{S} = \vec{E} \times \vec{H}$$

$$\vec{H} = (\hat{k} \times \vec{E})/Z \qquad \text{where } Z = \sqrt{\mu/\varepsilon}$$

- $\vec{H}$ refers to the magnetic field due to the light, not including any induced magnetic fields in the presence of matter.
- *Coherent* light has the same phase over macroscopic distances and time
- *Polarized* light has the electric field aligned over macroscopic distances and time

Polarization

Sources of un-polarized light

- Hot atoms transfer kinetic energy to electrons randomly
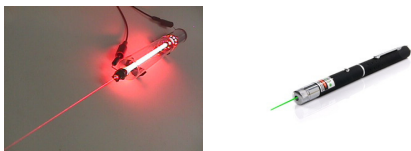- Electrons randomly de-excite, emitting incoherent light – uncorrelated in phase and polarization

Polarization

Sources of polarized light

- Lasers produce light by stimulated emission
  - A photon causes an excited atom to emit another photon
  - The photon is emitted in phase and with the same polarization
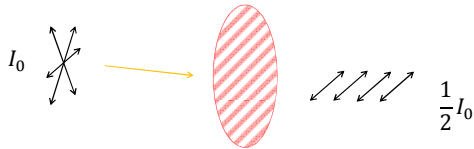- The resulting beam is highly coherent and polarized

Polarization by Absorption

- A polarizer absorbs the component with $\vec{E}$ oriented along a particular axis.
- The light that emerges is linearly polarized along the perpendicular axis.
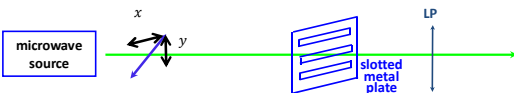- If the light is initially un-polarized, half the light is absorbed.

I_0 $\frac{1}{2}I_0$

Example with Microwaves

x LP

microwave source y slotted metal plate

- The electric field in the x-direction induces currents in the metal plate and loses energy:
  - Horizontally polarized microwaves are absorbed
- No current can flow in the y-direction because of the slots
  - Vertically polarized microwaves are transmitted
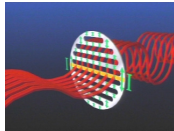
Polarization by Reflection

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- Reflected light is preferentially polarized
- The other component must be transmitted
- Transmission and reflection coefficients must depend on the polarization

Boundary Conditions

- In the presence of matter, the components electric and magnetic fields perpendicular to a surface change abruptly
- The component parallel to a surface is the same on both sides.
- Summary:
  - Perpendicular to surface
  $$\varepsilon_1 E_{1\perp} = \varepsilon_2 E_{2\perp}$$
  $$\mu_1 H_{1\perp} = \mu_2 H_{2\perp}$$
  - Parallel to surface
  $$E_{1\parallel} = E_{2\parallel}$$
  $$H_{1\parallel} = H_{2\parallel}$$
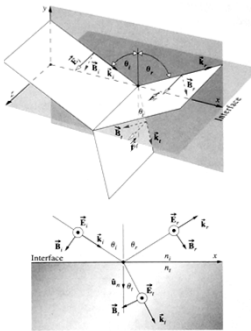
Reflection From a Surface

First case:

$\vec{E}$ is parallel to the surface…
$$E_i + E_r = E_t$$

$\vec{H}$ has components parallel and perpendicular to the surface
$$H_{\parallel i} + H_{\parallel r} = H_{\parallel t}$$

But $H_{\parallel} = \vec{H} \cdot \hat{\tau}$…

Reflection From a Surface

$\vec{E}$ **is perpendicular to** $\hat{n}$
$$\hat{\tau} = \frac{\vec{n} \times \hat{E}}{|\vec{E}|}$$

$\vec{H}$ can be written
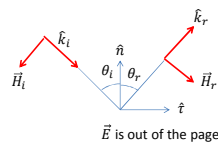$$\vec{H} = \frac{\hat{k} \times \vec{E}}{Z}$$

So we can write

$$\vec{H}_i \cdot \hat{\tau} = \frac{1}{ZE_i}\left(\hat{k} \times \vec{E}_i\right) \cdot \left(\vec{n} \times \hat{E}_i\right)$$
$$= \frac{E_i}{Z}\left(\hat{k} \cdot \hat{n}\right) = -\frac{E_i}{Z}\cos\theta_i$$

$\vec{E}$ is out of the page

Likewise,

$$\vec{H}_r \cdot \hat{\tau} = \frac{E_r}{Z}\cos\theta_r$$

$\vec{E}$ perpendicular to $\hat{n}$

- Boundary condition for $\vec{H}$:
$$H_{\|i} + H_{\|r} = H_{\|t}$$
- Previous results:
$$H_{i\|} = -\frac{E_i}{Z_1}\cos\theta_i \quad H_{r\|} = \frac{E_r}{Z_1}\cos\theta_r = \frac{E_r}{Z_1}\cos\theta_i$$

$$\frac{-E_i\cos\theta_i + E_r\cos\theta_i}{Z_1} = \frac{-E_t\cos\theta_t}{Z_2}$$
$$E_i + E_r = E_t$$
- Two equations in two unknowns…

$\vec{E}$ perpendicular to $\hat{n}$

$$\frac{-E_i\cos\theta_i + E_r\cos\theta_i}{Z_1} = \frac{-E_t\cos\theta_t}{Z_2}$$
$$\frac{E_i\cos\theta_t + E_r\cos\theta_t}{Z_2} = \frac{E_t\cos\theta_t}{Z_2}$$
- Solve for E_r/E_i:
$$\frac{E_r}{E_i} = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}$$
- Solve for E_t/E_i:
$$\frac{E_t}{E_i} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t}$$

Reflection From A Surface

$\vec{H}$ parallel to surface
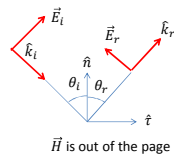$$H_i + H_r = H_t$$
$$E_{\|i} + E_{\|r} = E_{\|t}$$
$$E_i\cos\theta_i - E_r\cos\theta_r = E_t\cos\theta_t$$
$$\frac{E_i}{Z_1} + \frac{E_r}{Z_1} = \frac{E_t}{Z_2}$$
- Two equations in two unknowns…

$\vec{H}$ is out of the page

$\vec{H}$ perpendicular to $\hat{n}$

$$\frac{E_i \cos\theta_i - E_r \cos\theta_i}{Z_2} = \frac{E_t \cos\theta_t}{Z_2}$$

$$\frac{E_i \cos\theta_t + E_r \cos\theta_t}{Z_1} = \frac{E_t \cos\theta_t}{Z_2}$$

- Solve for E_r/E_i:
$$\frac{E_r}{E_i} = \frac{Z_1\cos\theta_i - Z_2\cos\theta_t}{Z_1\cos\theta_i + Z_2\cos\theta_t}$$
- Solve for E_t/E_i:
$$\frac{E_t}{E_i} = \frac{2Z_1\cos\theta_i}{Z_1\cos\theta_i + Z_2\cos\theta_t}$$

Fresnel's Equations

- In most dielectric media, $\mu_1 = \mu_2$ and therefore
$$\frac{Z_1}{Z_2} = \sqrt{\frac{\mu_1\varepsilon_2}{\mu_2\varepsilon_1}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1} = \frac{\sin\theta_i}{\sin\theta_t}$$
- After some trigonometry...
$$\left(\frac{E_r}{E_i}\right)_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} \qquad \left(\frac{E_r}{E_i}\right)_\parallel = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$\left(\frac{E_t}{E_i}\right)_\perp = \frac{(n_1/n_2)\sin(2\theta_i)}{\sin(\theta_i + \theta_t)} \quad \left(\frac{E_t}{E_i}\right)_\parallel = \frac{2\cos(\theta_i)\sin(\theta_t)}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)}$$

For $\vec{E}$ perpendicular and parallel to **plane of incidence**.
(not the same as perpendicular or parallel to the surface)

Normal Incidence

- At normal incidence, $\cos\theta = 1$ there really is no component parallel to the plane of incidence.
- In this case
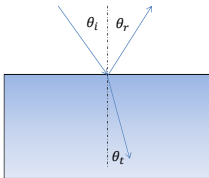$$\left(\frac{E_r}{E_i}\right)_\perp = \frac{n_1 - n_2}{n_1 + n_2}$$
$$\left(\frac{E_t}{E_i}\right)_\perp = \frac{2n_2}{n_1 + n_2}$$
- This what we arrived at previously using only the difference in the speed of light in the two materials.
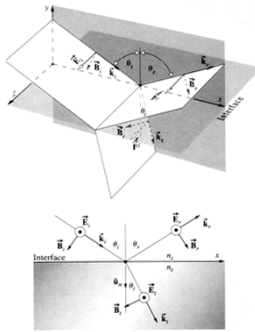
Application of Fresnel's Equations

- Unpolarized light in air ($n = 1$) is incident on a surface with index of refraction $n' = 1.5$ at an angle $\theta_i = 30°$
- What are the magnitudes of the electric field components of the reflected light?

First calculate the angles of reflection and refraction…

$$\boldsymbol{\theta_r = \theta_i = 30°}$$
$$\sin\theta_i = n' \sin\theta_t$$
$$\sin\theta_t = \frac{\sin 30°}{1.5} = 0.333$$
$$\boldsymbol{\theta_t = 19.5°}$$

Application of Fresnel's Equations

- Component of $\vec{E}$ parallel to the surface is perpendicular to the plane of incidence

$$\left(\frac{E_r}{E_i}\right)_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$
$$= -\frac{\sin(30° - 19.5°)}{\sin(30° + 19.5°)}$$
$$= -0.240$$

Application of Fresnel's Equations

- Component of $\vec{E}$ perpendicular to the surface is parallel to the plane of incidence

$$\left(\frac{E_r}{E_i}\right)_\parallel = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$
$$= \frac{\tan(30° - 19.5°)}{\tan(30° + 19.5°)}$$
$$= 0.158$$

- Reflected light is preferentially polarized parallel to the surface.

- Component of $\vec{E}$ perpendicular to the surface is parallel to the plane of incidence

$$\left(\frac{E_r}{E_i}\right)_\parallel = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$
$$= \frac{\tan(30° - 19.5°)}{\tan(30° + 19.5°)}$$
$$= 0.158$$

- Reflected light is preferentially polarized parallel to the surface.

Reflected Components

- Since $\theta_t < \theta_i$ the component perpendicular to the plane of incidence is always negative:
$$\left(\frac{E_r}{E_i}\right)_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

- The component parallel to the plane of incidence could be positive or negative:
$$\left(\frac{E_r}{E_i}\right)_\parallel = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

- What happens when $\theta_i + \theta_t = 90°$?
  - Can this happen? Sure… check when $\theta_i \rightarrow 90°$.
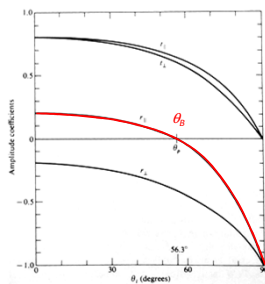
Brewster's Angle

$\tan \theta \rightarrow \infty$ as $\theta \rightarrow 90°$ while $\tan(\theta_i - \theta_t)$ remains finite.

Therefore,
$$\left(\frac{E_r}{E_i}\right)_\parallel \rightarrow 0$$

Brewster's angle, θ_B is the angle of incidence for which $r_\parallel \rightarrow 0$.

Brewster's Angle

- Brewster's angle can be calculated from the relation $\theta_i + \theta_t = 90°$
- We can always calculate θ_t using Snell's law
- Good assignment question:

  Show that $\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right)$

- Light reflected from a surface at an angle θ_B will be linearly polarized parallel to the surface (perpendicular to the plane of incidence)

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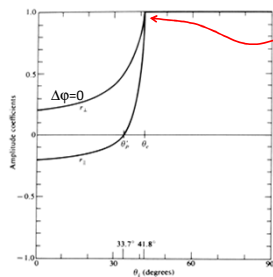
Total Internal Reflection

- Consider the other case when $n_i > n_t$, for example, glass to air:

At some incidence angle (critical angle θ_c) everything is reflected (and nothing transmitted).

It can be shown that for any angle larger than θ_c no waves are transmitted into media: **total internal reflection**.

No phase shift upon reflection.

Reflected Intensity

- Remember that the *intensity* (*irradiance*) is related to the energy carried by light:
$$I = \epsilon v \langle E^2 \rangle_T$$
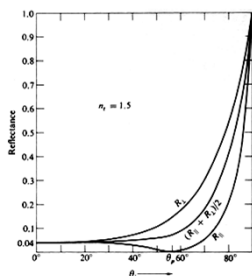(averaged over some time $T \gg 1/f$)

- Reflectance is defined as
$$R_\perp = (r_\perp)^2 = \frac{\sin^2(\theta_i - \theta_t)}{\sin^2(\theta_i + \theta_t)}$$
$$R_\parallel = (r_\parallel)^2 = \frac{\tan^2(\theta_i - \theta_t)}{\tan^2(\theta_i + \theta_t)}$$

- Unpolarized reflectance:
$$R = \frac{1}{2}(R_\perp + R_\parallel)$$

Reflected Intensity

- How polarized is the reflected light?
- Degree of polarization:
$$V = I_p / I_{total}$$
- Measured using an analyzing polarizer

I_r → $I_{analyzed}$

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$