PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 3 – French, Chapter 1

Spring 2016 Semester

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Announcements

1. Assignment #1 should be straight forward
   - You can download it from the course web page
   - Ask questions in class if you are completely stuck
   - Make use of office hours if you are still completely stuck…
2. No class on Monday…

   http://www.purdue.edu/diversity-inclusion/mlk/
3. Favorite quote from Martin Luther King, Jr:

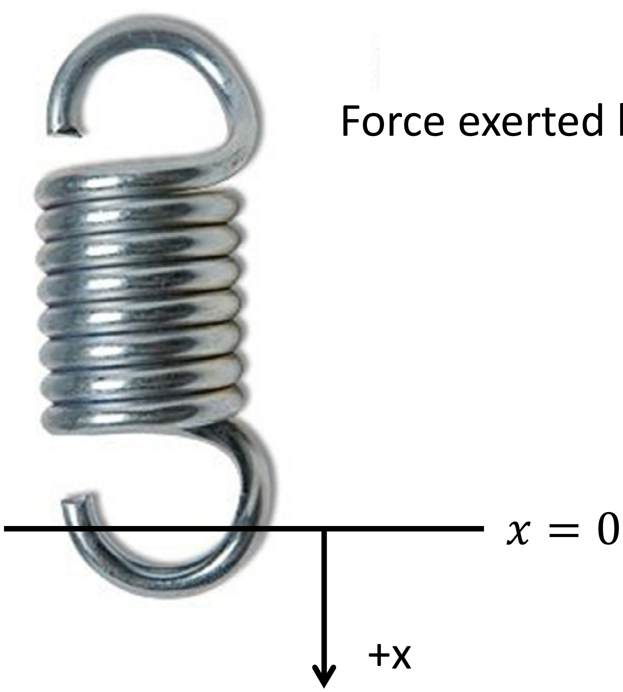
Nothing in all the world
is more dangerous
than sincere ignorance
and conscientious stupidity.

The Mass/Spring System

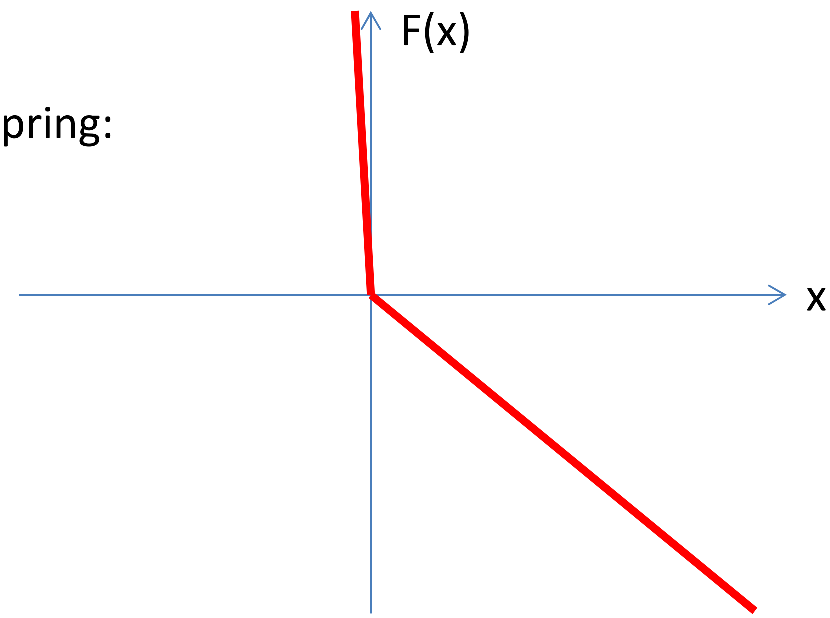
- We've been discussing a mass attached to a spring:
  - Force acting ON the mass:
$$F(x) = -k\,x$$
  - Acceleration due to the force:
$$F(x) = m\frac{d^2x}{dt^2}$$
- So far, we haven't said much about the coordinate system we were using.
- This is because Hooke's law, as written, defines both the origin ($x = 0$ when the force vanishes).
- We didn't specify what direction +x was, but the solution would be consistent with the initial conditions.

The Mass/Spring System

- We also ignored other forces on the mass, (namely gravity) which we know exists.
- Furthermore, we can see that $F(x) = -k\ x$ will not describe the spring when it is fully compressed.

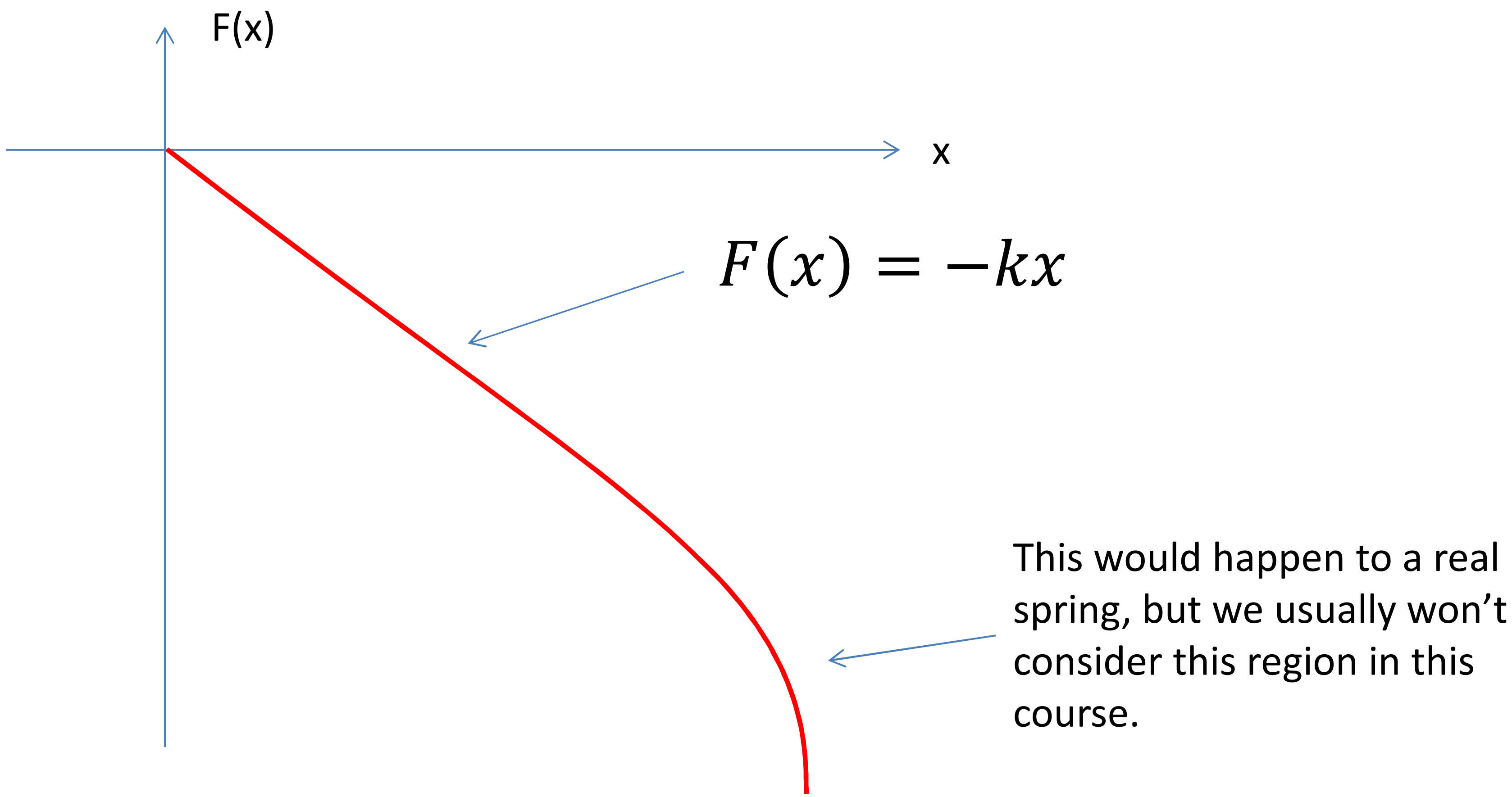


Force exerted by the spring:



The Mass/Spring System

- We are also usually interested in displacements where the force is described by Hooke's law.



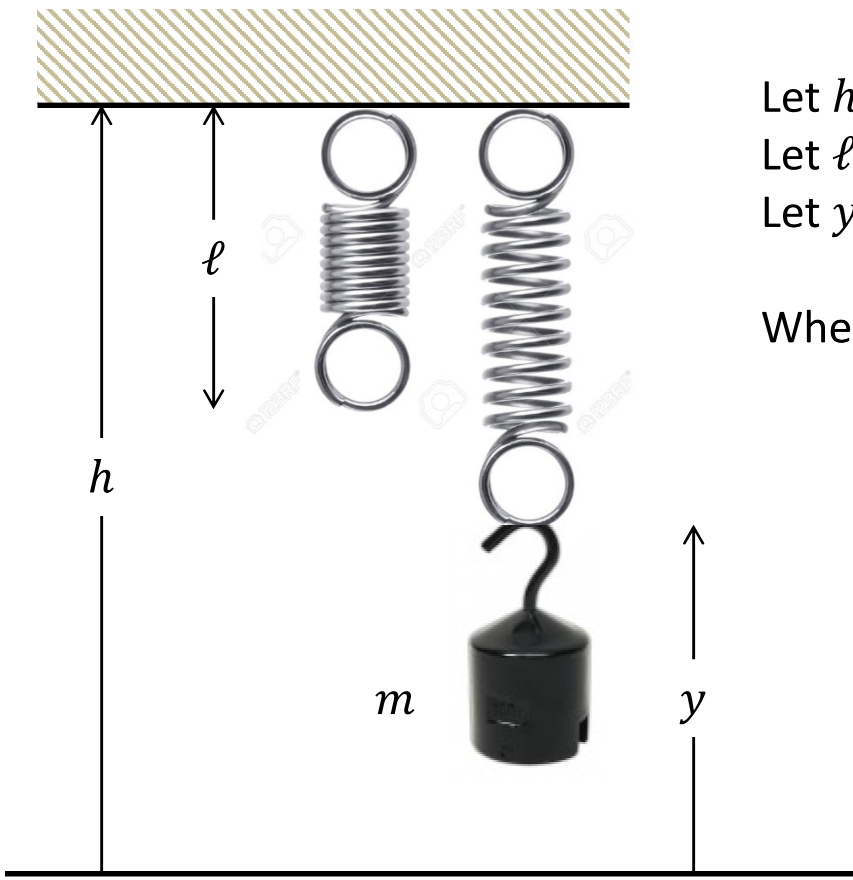
The Mass/Spring System

- Let's analyze a more realistic mass/spring problem:

A spring with uncompressed length ℓ is attached to a support which is at a height h. A mass m is attached to the other end of the spring. If the mass is initially displaced so that it is at a height y_0, what will the resulting motion be when it is released from rest?

Mass/Spring System

- You will have to draw a diagram:

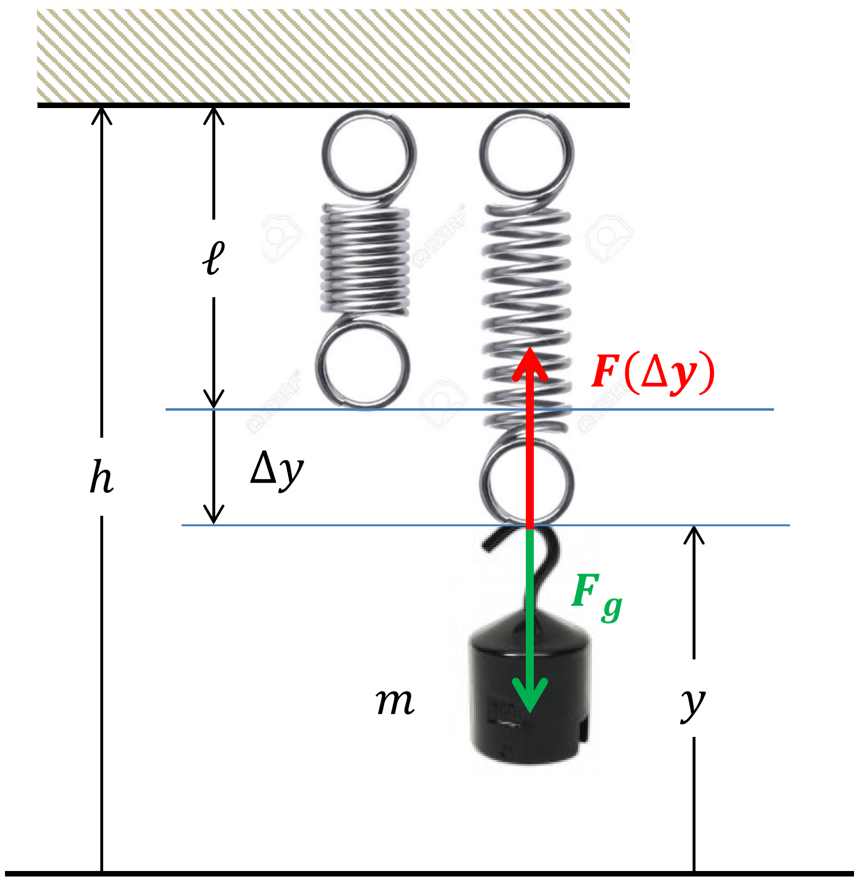


Let h be the height of the support.
Let ℓ be the uncompressed length of the spring.
Let y represent the height of the mass.

When the spring is uncompressed, $y = h - \ell$.

Mass/Spring System

- Identify the forces:



For a given y, how much has the spring been stretched?

$$\Delta y = h - \ell - y$$

We assume that the force is, at all times, described by Hooke's law:

$$F(\Delta y) = +k\,\Delta y$$

There is also the constant force of gravity:

$$F_g = -mg$$

Mass/Spring System

- Sum of the forces:

$$\sum F_y = F(\Delta y) + F_g = k\Delta y \ - mg$$

- But recall that $\Delta y = h - \ell - y$…

$$\sum F_y = k(h - \ell - y) - mg$$

$$= \underbrace{k(h - \ell) - mg}_{\text{This is just a constant.}} - ky$$

Try to stay organized… keep all constant terms together!

Mass/Spring System

- Newton's second law:

$$\sum F_y = m\,\frac{d^2y}{dt^2}$$

- This is the differential equation we must solve:

$$m\,\frac{d^2y}{dt^2} - F_y = 0$$

$$m\,\frac{d^2y}{dt^2} - k(h-\ell) + mg + ky = 0$$

$$\ddot{y} + \omega^2 y = \frac{k(h-\ell) - mg}{m}$$

Mass/Spring System

$$\ddot{y} + \omega^2 y = \frac{k}{m}(h - \ell) - g$$

But wait! This is NOT of the form

$$\ddot{y} + \omega^2 y = 0$$

- *Therefore, the solutions are not going to be of the form*

$$y(t) = \boldsymbol{A}\,\cos(\omega t + \boldsymbol{\varphi})$$

- But then what are they?

Homogeneous Equation

- This is called the homogeneous equation:
$$\ddot{y} + \omega^2 y = 0$$
- The general solutions to the homogeneous equation are of the form:
$$y(t) = \boldsymbol{A}\cos(\omega t + \boldsymbol{\varphi})$$

Nonhomogeneous Equation

$$\ddot{y} + \omega^2 y = \frac{k(h - \ell) - mg}{m}$$

- A solution to the non-homogeneous equation is called a “particular solution”. Can we find one?

$$y = \frac{k(h - \ell) - mg}{\omega^2 m} = (h - \ell) - \frac{mg}{k}$$

- This solves the nonhomogeneous equation because it is just a constant, so

$$\frac{d^2 y}{dt^2} = 0$$

Nonhomogeneous Equation

- A general solution to the nonhomogeneous equation is the sum of a general solution to the homogeneous equation and a particular solution to the nonhomogeneous equation.

$$y(t) = \boldsymbol{A}\cos(\omega t + \boldsymbol{\varphi}) + (h - \ell) - \frac{mg}{k}$$

- Now we can solve for the constants of integration using the initial conditions.

Nonhomogeneous Equation

$$y(t) = \boldsymbol{A}\cos(\omega t + \boldsymbol{\varphi}) + (h - \ell) - \frac{mg}{k}$$

- The initial velocity is:

$$\left.\frac{dy}{dt}\right|_{t=0} = -A\omega\sin(\boldsymbol{\varphi}) = 0$$

- Therefore, $\varphi = 0$.
- The initial displacement is

$$y(0) = \boldsymbol{A} + (h - \ell) - \frac{mg}{k} = y_0$$

- Therefore, $A = y_0 - (h - \ell) + mg/k$.

Another way to look at it…

- We expect the mass to oscillate about an equilibrium position.
- What is the equilibrium displacement, $\bar{y}$?
- When the mass is at rest,

$$\frac{dy}{dt} = 0 \text{ and } \frac{d^2y}{dt^2} = 0$$

- The spring force and the force of gravity must cancel:

$$k(h - \ell - \bar{y}) = mg$$

$$\bar{y} = (h - \ell) - \frac{mg}{k}$$

Another way to look at it…

- What is the net force due to a small displacement, u, away from equilibrium?

$$F(u) = -k\,u$$

- Differential equation for u:

$$m\frac{d^2u}{dt^2} = -k\,u$$

$$\ddot{u} + \omega^2 u = 0$$

- This we know how to solve:

$$u(t) = \boldsymbol{A}\cos(\omega t + \boldsymbol{\varphi})$$

- Absolute position of the mass is

$$y(t) = \overline{y} + u = \boldsymbol{A}\cos(\omega t + \boldsymbol{\varphi}) + (h - \ell) - \frac{mg}{k}$$

Observations

- The frequency, ω, does not depend on
  - The initial conditions
  - Additional constant forces
- The mass oscillates about an equilibrium position which can be calculated using statics.
  - Just algebra, since all derivatives vanish
- Coordinate transformations can be very useful
  - The nonhomogeneous equation can be transformed to a homogeneous equation by adding a constant offset to the displacement

Deviations from Hooke's Law

- Not all forces are perfectly linear in displacement.
- Sometimes they are approximately linear for only small displacements, u, about an equilibrium position.
  - At the equilibrium position, $u = 0$, the force vanishes
    $$F(0) = 0$$
  - An arbitrary force can be described by a power series
    $$F(u) = a_1 u + a_2 u^2 + a^3 u^3 + \ ...$$
    (notice that there is no constant a_0 term)
  - How can we determine the coefficients?

Deviations from Hooke's Law

- Differentiate both sides, evaluate for $u = 0$:

$$\left.\frac{dF}{du}\right|_{u=0} = \left(a_1 + 2a_2 u + 3a_3 u^2 + \cdots\right)\Big|_{u=0}$$

$$a_1 = F'(0)$$

- Hooke's law: $F(u) = -k\,u$
- Approximation for small displacements:

$$F(u) \approx a_1\,u$$

- The "effective" spring constant for small displacements is

$$k = -\left.\frac{dF}{du}\right|_{u=0}$$

Potential Energy

- Recall from mechanics that for conservative forces, we can write
$$\vec{F} = -\nabla V$$
- In just one dimension this is
$$F = -\frac{dV}{dx}$$
- Write the potential energy as a power series:
$$V(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

Potential Energy

- The force is:

$$F = -\frac{dV}{dx} = -b_1 - 2b_2x \; - \cdots$$

- The force vanishes when $dV/dx = 0$
  - This is the equilibrium position, $\bar{x}$.
- The effective spring constant is

$$k = -\frac{dF}{dx} = \frac{d^2V}{dx^2}$$

(This must be positive)