PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 29 – Geometric Optics

Spring 2016 Semester

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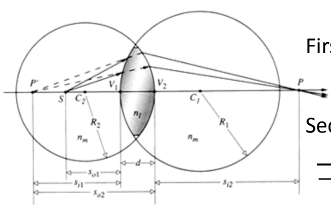
Thin Lens Equation



First surface:

$$\frac{n_m}{s_{o1}} + \frac{n_l}{s_{i1}} = \frac{n_l - n_m}{R_1}$$

Second surface:

$$\frac{n_l}{-s_{i1} + d} + \frac{n_m}{s_{i2}} = \frac{n_m - n_l}{R_2}$$

Add these equations and simplify using $n_m = 1$ and $d \to 0$:

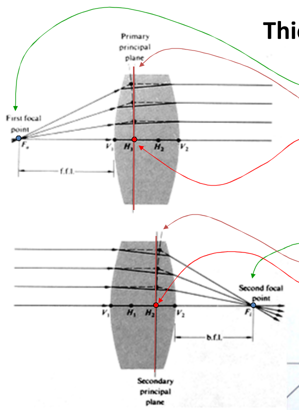
$$\frac{1}{s_o} + \frac{1}{s_i} = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

(Thin lens equation)

Thick Lenses

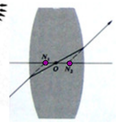
- Eliminate the intermediate image distance, s_{i1}
- Focal points:
  - Rays passing through the focal point are refracted parallel to the optical axis by both surfaces of the lens
  - Rays parallel to the optical axis are refracted through the focal point
  - For a thin lens, we can draw the point where refraction occurs in a common plane
  - For a thick lens, refraction for the two types of rays can occur at different planes

Thick Lens: definitions



First focal point (front focal length)
Primary principal plane
First principal point

Second focal point (back focal length)
Secondary principal plane
Second principal point



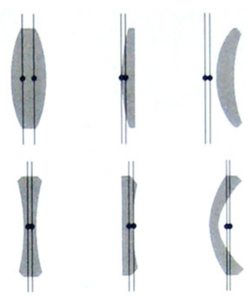
Nodal points

If media on both sides has the same *n*, then: $N_1=H_1$ and $N_2=H_2$

$F_o\ F_i\ H_1\ H_2\ N_1\ N_2$ - **cardinal points**

Thick Lens: Principal Planes

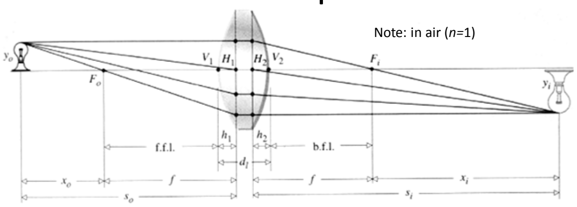
Principal planes can lie outside the lens:



Thick Lenses and Principal Planes

- For a single refracting surface, we measured s_i and s_o with respect to the vertex (ie, the surface of the lens)
- For a thick lens, we need to define s_i and s_o with respect to the principal planes.
- We need to calculate where they are, but it makes the algebra simpler.
- We are not going to derive the following formula…

Thick Lens: equations



$$\frac{1}{s_o}+\frac{1}{s_i}=\frac{1}{f}$$ $$x_o x_i = f^2$$

effective focal length:

$$\frac{1}{f}=(n_l-1)\left[\frac{1}{R_1}-\frac{1}{R_2}+\frac{(n_l-1)d_l}{n_l R_1 R_2}\right]$$

Principal planes: $h_1=-\frac{f(n_l-1)d_l}{n_l R_2}$ $h_2=-\frac{f(n_l-1)d_l}{n_l R_1}$

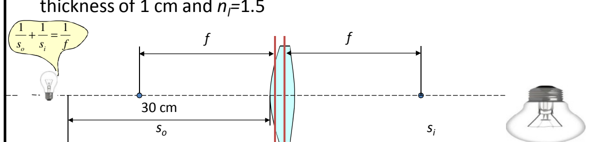
Magnification: $M_T \equiv \frac{y_i}{y_o}=-\frac{s_i}{s_o}=-\frac{x_i}{f}=-\frac{f}{x_o}$

Thick Lens Calculations

1. Calculate focal length
$$\frac{1}{f}=(n-1)\left[\frac{1}{R_1}-\frac{1}{R_2}+\frac{(n-1)d}{nR_1R_2}\right]$$
2. Calculate positions of principal planes
$$h_1=-\frac{f(n-1)d}{nR_2}$$
$$h_2=-\frac{f(n-1)d}{nR_1}$$
3. Calculate object distance, s_o, measured from principal plane
4. Calculate image distance:
$$\frac{1}{s_o}+\frac{1}{s_i}=\frac{1}{f}$$
5. Calculate magnification, $m_T=-s_i/s_o$

Thick Lens: example

Find the image distance for an object positioned 30 cm from the vertex of a double convex lens having radii 20 cm and 40 cm, a thickness of 1 cm and n_l=1.5



$$\frac{1}{f}=(n_l-1)\left[\frac{1}{R_1}-\frac{1}{R_2}+\frac{(n_l-1)d_l}{n_l R_1 R_2}\right]=0.5\left[\frac{1}{20}-\frac{1}{-40}-\frac{0.5\cdot 1}{1.5\cdot 20\cdot 40}\right]\frac{1}{\text{cm}}$$

$f=26.8\,\text{cm}$

$$h_1=-\frac{26.8\cdot 0.5\cdot 1}{-40\cdot 1.5}\text{cm}=0.22\text{cm}$$

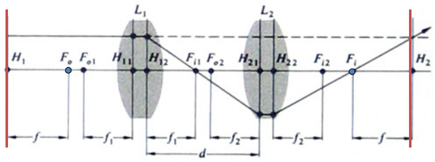
$$h_2=-\frac{26.8\cdot 0.5\cdot 1}{20\cdot 1.5}\text{cm}=-0.44\text{cm}$$

$s_o=30\text{cm}+0.22\text{cm}=30.22\,\text{cm}$

$$\frac{1}{30.22\text{cm}}+\frac{1}{s_i}=\frac{1}{26.8\text{cm}}$$

$s_i=238\,\text{cm}$

Compound Thick Lens



Can use two principal points (planes) and effective focal length f to describe propagation of rays through any compound system

Note: any ray passing through the first principal plane will emerge at the same height at the second principal plane

For 2 lenses (above): $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$ $\quad \overline{H_{11}H_1} = fd/f_2$, $\overline{H_{22}H_2} = fd/f_1$

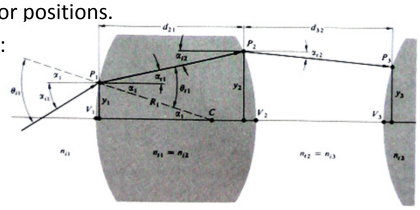
Example: page 246

Ray Tracing

- Even the thick lens equation makes approximations and assumptions
  - Spherical lens surfaces
  - Paraxial approximation
  - Alignment with optical axis
- The only physical concepts we applied were
  - Snell's law: $n_i \sin\theta_i = n_t \sin\theta_t$
  - Law of reflection: $\theta_t = \theta_i$ (in the case of mirrors)
- Can we do better? Can we solve for the paths of the rays exactly?
  - Sure, no problem! But it is a lot of work.
  - Computers are good at doing lots of work (without complaining)

Ray Tracing

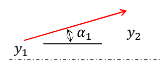
- We will still make the assumptions of
  - Paraxial rays
  - Lenses aligned along optical axis
- We will make no assumptions about the lens thickness or positions.
- Geometry:



Ray Tracing

- At a given point along the optical axis, each ray can be uniquely represented by two numbers:
  - Distance from optical axis, y_i
  - Angle with respect to optical axis, α_i
- If the ray does not encounter an optical element its distance from the optical axis changes according to the *transfer equation*:

$$y_2 = y_1 + d_1\alpha_1$$



  - This assumes the paraxial approximation $\sin\alpha_1 \approx \alpha_1$

Ray Tracing

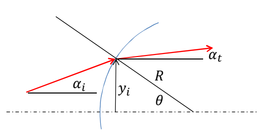
- At a given point along the optical axis, each ray can be uniquely represented by two numbers:
  - Distance from optical axis, y_i
  - Angle with respect to optical axis, α_i
- When the ray encounters a surface of a material with a different index of refraction, its angle will change according to the *refraction equation*:

$$n_{t1}\alpha_{t1} = n_{i1}\alpha_{i1} - D_1 y_1$$
$$D_1 = \frac{n_{t1} - n_{i1}}{R_1}$$

  - Also assumes the paraxial approximation

Ray Tracing

- Geometry used for the refraction equation:

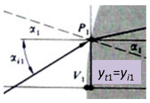


$$\sin\theta = \frac{y_i}{R} \approx \theta$$
$$\theta_i = \alpha_i + \theta = \alpha_i + y_i/R$$
$$\theta_t = \alpha_t + \theta = \alpha_t + y_i/R$$
$$n_i\theta_i = n_t\theta_t$$
$$n_i\alpha_i + \frac{n_i y_i}{R} = n_t\alpha_t + \frac{n_t y_i}{R}$$
$$n_t\alpha_t = n_i\alpha_i - \left(\frac{n_t - n_i}{R}\right)y_i$$

Matrix Treatment: Refraction

At any point of space need 2 parameters to fully specify ray: distance from axis (y) and inclination angle (α) with respect to the optical axis. Optical element changes these ray parameters.

Refraction:



$$n_{t1}\alpha_{t1} = n_{i1}\alpha_{i1} - \mathrm{D}_1 y_{i1}$$
$$y_{t1} = 0 \cdot n_{i1}\alpha_{i1} + y_{i1}$$

note: paraxial approximation

Reminder:
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \alpha \\ y \end{pmatrix} \equiv \begin{pmatrix} A\alpha + By \\ C\alpha + Dy \end{pmatrix}$$

Equivalent matrix representation:

$$\begin{pmatrix} n_{t1}\alpha_{t1} \\ y_{t1} \end{pmatrix} = \begin{pmatrix} 1 & -\mathrm{D}_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} n_{i1}\alpha_{i1} \\ y_{i1} \end{pmatrix}$$

$\equiv \mathbf{r}_{i1}$ - input ray

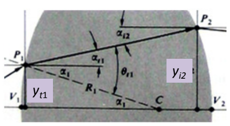
$\equiv \mathbf{R}_1$ - ***refraction matrix***

$\equiv \mathbf{r}_{t1}$ - output ray

$$\mathbf{r}_{t1} = \mathbf{R}_1 \mathbf{r}_{i1}$$

Matrix: Transfer Through Space

Transfer:



$$n_{i2}\alpha_{i2} = n_{t1}\alpha_{t1} + 0 \cdot y_{t1}$$
$$y_{i2} = d_{21} \cdot \alpha_{t1} + y_{t1}$$

Equivalent matrix presentation:

$$\begin{pmatrix} n_{i2}\alpha_{i2} \\ y_{i2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d_{21}/n_{t1} & 1 \end{pmatrix}\begin{pmatrix} n_{t1}\alpha_{t1} \\ y_{t1} \end{pmatrix}$$

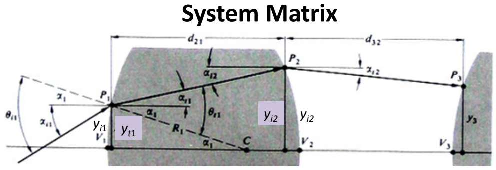
$\equiv \mathbf{r}_{t1}$ - input ray

$\equiv \mathbf{T}_{21}$ - ***transfer matrix***

$\equiv \mathbf{r}_{i2}$ - output ray

$$\mathbf{r}_{i2} = \mathbf{T}_{21} \mathbf{r}_{t1}$$

System Matrix



$\mathbf{r}_{i1} \rightarrow \mathbf{r}_{t1} \rightarrow \mathbf{r}_{i2} \rightarrow \mathbf{r}_{t2} \rightarrow \mathbf{r}_{i3} \rightarrow \mathbf{r}_{t3}$

$\mathbf{R}_1$ $\mathbf{T}_{21}$ $\mathbf{T}_{32}$ $\mathbf{R}_3$

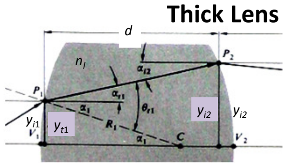
$$\mathbf{r}_{t1} = \mathbf{R}_1 \mathbf{r}_{i1}$$
$$\mathbf{r}_{i2} = \mathbf{T}_{21} \mathbf{r}_{t1} = \mathbf{T}_{21} \mathbf{R}_1 \mathbf{r}_{i1}$$

Thick lens ray transfer: $\mathbf{r}_{t2} = \mathbf{R}_2 \mathbf{T}_{21} \mathbf{R}_1 \mathbf{r}_{i1}$

System matrix: $\mathbf{A} = \mathbf{R}_2 \mathbf{T}_{21} \mathbf{R}_1 \rightarrow \mathbf{r}_{t2} = \mathbf{A}\ \mathbf{r}_{i1}$

Can treat any system with single system matrix

Thick Lens Matrix



$$\mathbf{A} = \mathbf{R}_2\mathbf{T}_{21}\mathbf{R}_1$$

$$\mathbf{R} = \begin{pmatrix} 1 & -\mathcal{D} \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ d/n & 1 \end{pmatrix}$$

Reminder:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} Aa+Bc & Ab+Bd \\ Ca+Dc & Cb+Dd \end{pmatrix}$$

$$\frac{1}{f} = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\mathbf{A} = \begin{pmatrix} 1 & -\mathcal{D}_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ d_l/n_l & 1 \end{pmatrix}\begin{pmatrix} 1 & -\mathcal{D}_1 \\ 0 & 1 \end{pmatrix}$$

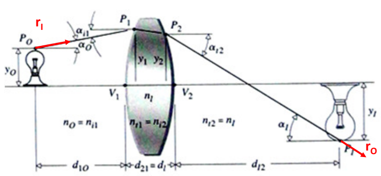
system matrix of thick lens

$$\mathbf{A} = \begin{pmatrix} 1 - \dfrac{\mathcal{D}_2 d_l}{n_l} & -\mathcal{D}_1 - \mathcal{D}_2 - + \dfrac{\mathcal{D}_1\mathcal{D}_2 d_l}{n_l} \\ \dfrac{d_l}{n_l} & 1 - \dfrac{\mathcal{D}_1 d_l}{n_l} \end{pmatrix}$$

For thin lens d_l=0

$$\mathbf{A} = \begin{pmatrix} 1 & -1/f \\ 0 & 1 \end{pmatrix}$$

Matrix Treatment: example



$$\mathbf{r}_I = \mathbf{T}_I\mathbf{A}_l\mathbf{T}_2\mathbf{r}_O$$

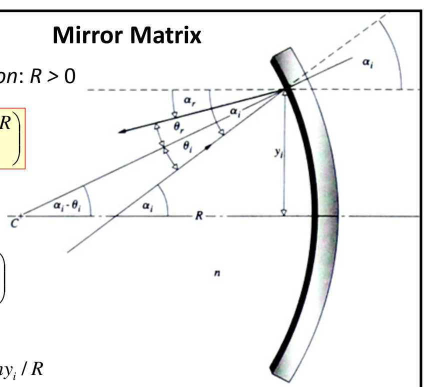
$$\begin{pmatrix} n_I\alpha_I \\ y_I \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d_{I2}/n_I & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ d_{1O}/n_O & 1 \end{pmatrix}\begin{pmatrix} n_O\alpha_O \\ y_O \end{pmatrix}$$

(Detailed example with thick lenses and numbers: page 250)

Mirror Matrix

Sign convention: $R > 0$

$$\mathbf{M} = \begin{pmatrix} -1 & 2n/R \\ 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} n\alpha_r \\ y_r \end{pmatrix} = \mathbf{M}\begin{pmatrix} n\alpha_i \\ y_i \end{pmatrix}$$

$$y_r = y_i$$

$$n\alpha_r = -n\alpha_i + 2ny_i/R$$

$$\alpha_r = -\alpha_i + 2y_i/R$$

Tray Tracing Example

- Transfer matrix (distance d in medium n_1):

$$\begin{pmatrix} n_2\alpha_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d/n_1 & 1 \end{pmatrix} \begin{pmatrix} n_1\alpha_1 \\ y_1 \end{pmatrix}$$

- Refraction matrix (spherical surface)

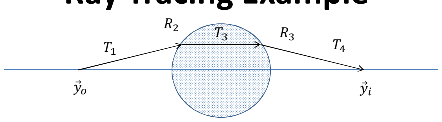
$$\begin{pmatrix} n_2\alpha_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -D \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n_1\alpha_1 \\ y_1 \end{pmatrix}$$

$$D = \frac{n_2 - n_1}{R}$$

- This example:

$$\vec{y}_o = T_5 R_4 T_3 R_2 T_1 \vec{y}_i$$

Ray Tracing Example



- Initial ray:

$$\vec{y}_o = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

- Final ray should cross the optical axis at a distance s_i from the second vertex.
- Multiply the matrices, solve for s_i…

Ray Tracing Example

- Use Mathematica…

```
In[10]:= (*  Propagateray to the image point *)
         yi = tm[si, 1].y4
```

Out[10]= $\left\{\left\{\alpha + \frac{(1-n)\, so\alpha}{r} - \frac{(-1+n)\left(so\alpha + \frac{2r\left(\alpha + \frac{(1-n)\, so\alpha}{r}\right)}{n}\right)}{r}\right\}, \left\{so\alpha + \frac{2r\left(\alpha + \frac{(1-n)\, so\alpha}{r}\right)}{n} + si\left(\alpha + \frac{(1-n)\, so\alpha}{r} - \frac{(-1+n)\left(so\alpha + \frac{2r\left(\alpha + \frac{(1-n)\, so\alpha}{r}\right)}{n}\right)}{r}\right)\right\}\right\}$

```
In[13]:= (*  The conditionfor an image is that the ray crosses the optical axis
          at the image point.  So we need to solve for si as a functionof so.  *)
         solution = Solve[yi[[2]] == 0, {si}]
```

Out[13]= $\left\{\left\{si \to \frac{r\,(2r + 2so - n\,so)}{-2r + nr - 2so + 2n\,so}\right\}\right\}$

Ray Tracing Example

- Use Mathematica…

In[10]:=
```
(* Propagate ray to the image point *)
yi = tm[si, 1].y4
```

Out[10]=
$$\left\{\left\{\alpha+\frac{(1-n)\,so\,\alpha}{r}-\frac{(-1+n)\left(so\,\alpha+\frac{2\,r\left(\alpha+\frac{(1-n)\,so\,\alpha}{r}\right)}{n}\right)}{r}\right\},\left\{so\,\alpha+\frac{2\,r\left(\alpha+\frac{(1-n)\,so\,\alpha}{r}\right)}{n}+si\left(\alpha+\frac{(1-n)\,so\,\alpha}{r}-\frac{(-1+n)\left(so\,\alpha+\frac{2\,r\left(\alpha+\frac{(1-n)\,so\,\alpha}{r}\right)}{n}\right)}{r}\right)\right\}\right\}$$

In[13]:=
```
(* The condition for an image is that the ray crosses the optical axis
 at the image point. So we need to solve for si as a function of so. *)
solution = Solve[yi[[2]] == 0, {si}]
```

Out[13]=
$$\left\{\left\{si\to\frac{r\,(2\,r+2\,so-n\,so)}{-2\,r+n\,r-2\,so+2\,n\,so}\right\}\right\}$$

Ray Tracing Example

- Use Mathematica…
  - Object position was at the focal point of the first refracting surface:

$$s_o = \frac{R}{n-1}$$

In[15]:=
```
FullSimplify[si /. solution /. {so → r / (n - 1)}]
```

Out[15]=
$$\left\{\frac{r}{-1+n}\right\}$$

- It works!