Purdue University Department of Physics

Physics 42200

Waves & Oscillations

Lecture 29 – Geometric Optics

Spring 2016 Semester

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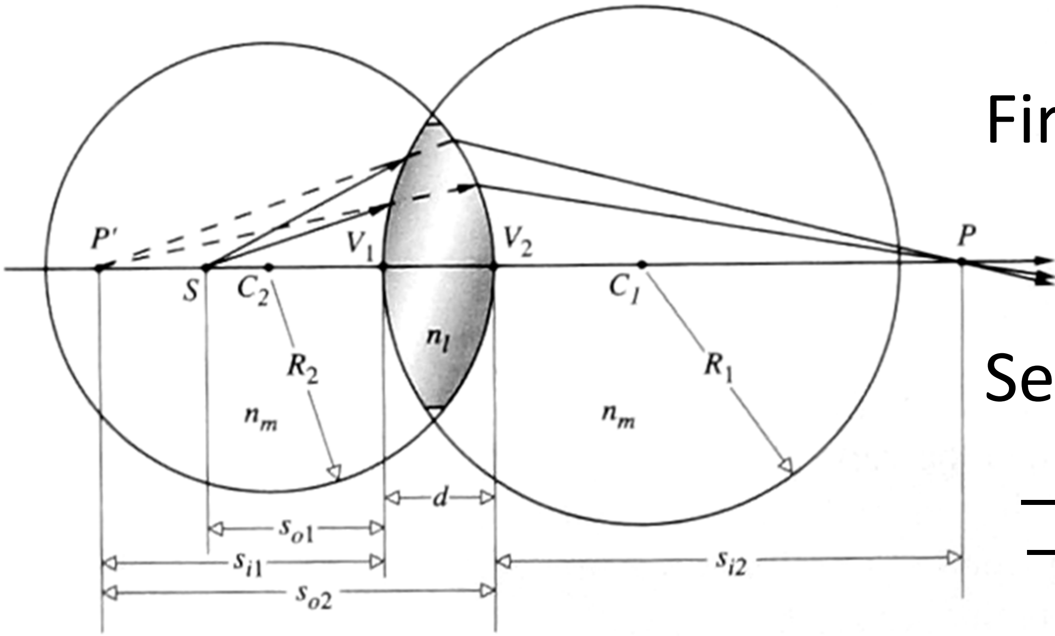
Thin Lens Equation



First surface:

$$\frac{n_m}{s_{o1}} + \frac{n_l}{s_{i1}} = \frac{n_l - n_m}{R_1}$$

Second surface:

$$\frac{n_l}{-s_{i1} + d} + \frac{n_m}{s_{i2}} = \frac{n_m - n_l}{R_2}$$

Add these equations and simplify using $n_m = 1$ and $d \to 0$:

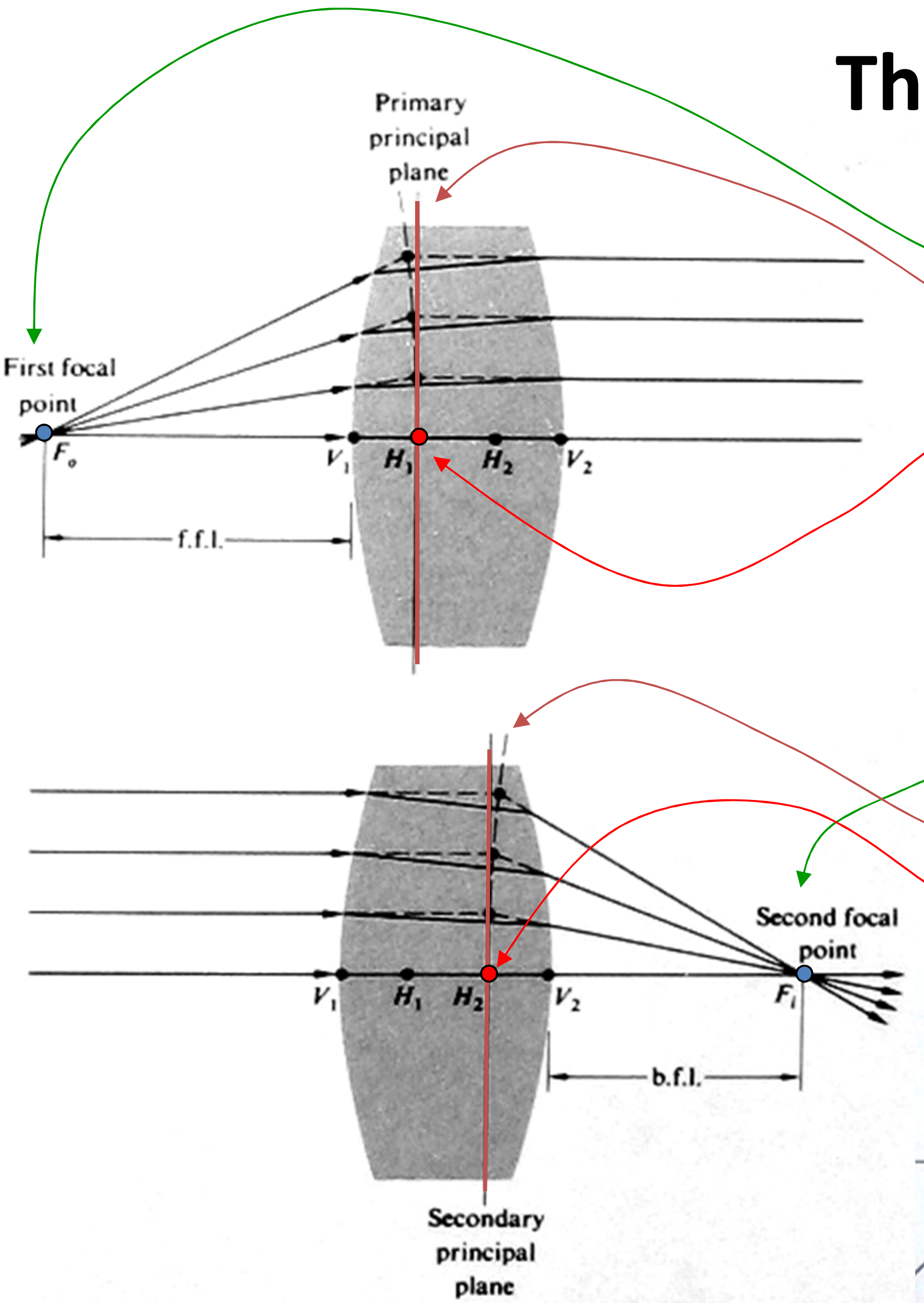
$$\frac{1}{s_o} + \frac{1}{s_i} = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

(Thin lens equation)

Thick Lenses

- Eliminate the intermediate image distance, s_{i1}
- Focal points:
  - Rays passing through the focal point are refracted parallel to the optical axis by both surfaces of the lens
  - Rays parallel to the optical axis are refracted through the focal point
  - For a thin lens, we can draw the point where refraction occurs in a common plane
  - For a thick lens, refraction for the two types of rays can occur at different planes

Thick Lens: definitions



First focal point (front focal length)

Primary principal plane

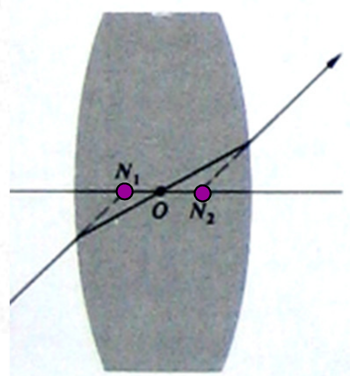
First principal point

Second focal point (back focal length)

Secondary principal plane

Second principal point

$F_o\ F_i\ H_1\ H_2\ N_1\ N_2$ - **cardinal points**

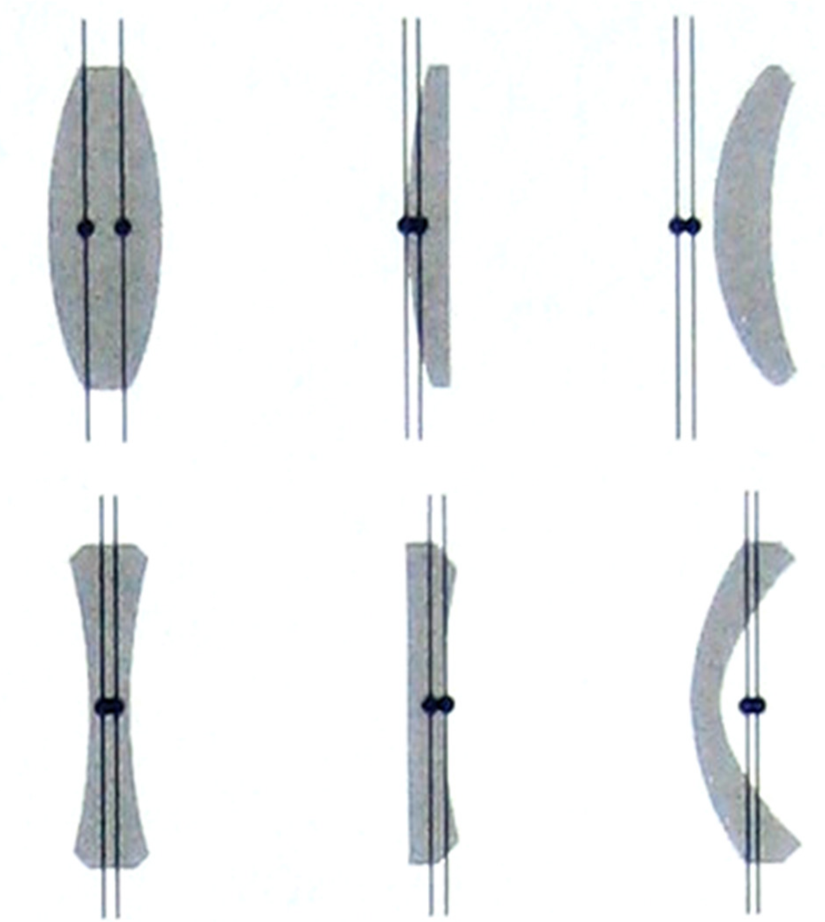


Nodal points

If media on both sides has the same n, then: $N_1 = H_1$ and $N_2 = H_2$

Thick Lens: Principal Planes

Principal planes can lie outside the lens:

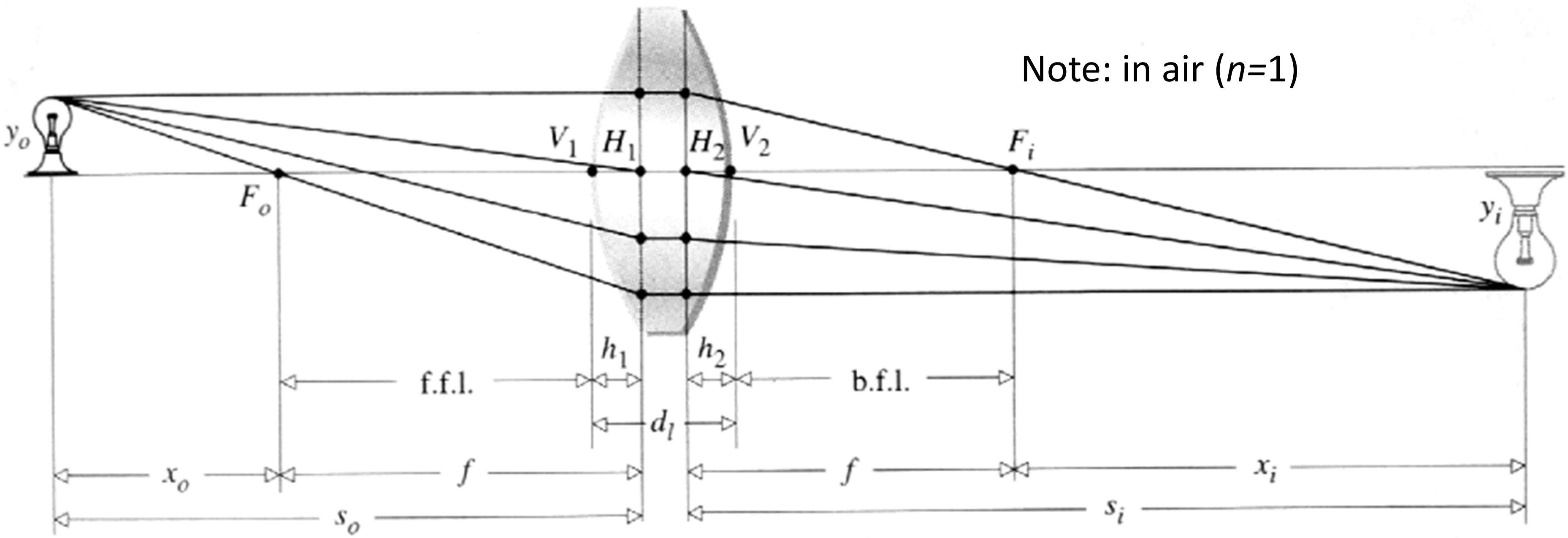


Thick Lenses and Principal Planes

- For a single refracting surface, we measured s_i and s_o with respect to the vertex (ie, the surface of the lens)
- For a thick lens, we need to define s_i and s_o with respect to the principal planes.
- We need to calculate where they are, but it makes the algebra simpler.
- We are not going to derive the following formula…

Thick Lens: equations

Note: in air (n=1)



$$\frac{1}{s_o}+\frac{1}{s_i}=\frac{1}{f}$$

$$x_o x_i = f^2$$

effective focal length:

$$\frac{1}{f}=(n_l-1)\left[\frac{1}{R_1}-\frac{1}{R_2}+\frac{(n_l-1)d_l}{n_l R_1 R_2}\right]$$

Principal planes:

$$h_1=-\frac{f(n_l-1)d_l}{n_l R_2}$$

$$h_2=-\frac{f(n_l-1)d_l}{n_l R_1}$$

Magnification:

$$M_T \equiv \frac{y_i}{y_o}=-\frac{s_i}{s_o}=-\frac{x_i}{f}=-\frac{f}{x_o}$$

Thick Lens Calculations

1. Calculate focal length

$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}\right]$$

2. Calculate positions of principal planes

$$h_1 = -\frac{f(n-1)d}{nR_2}$$

$$h_2 = -\frac{f(n-1)d}{nR_1}$$

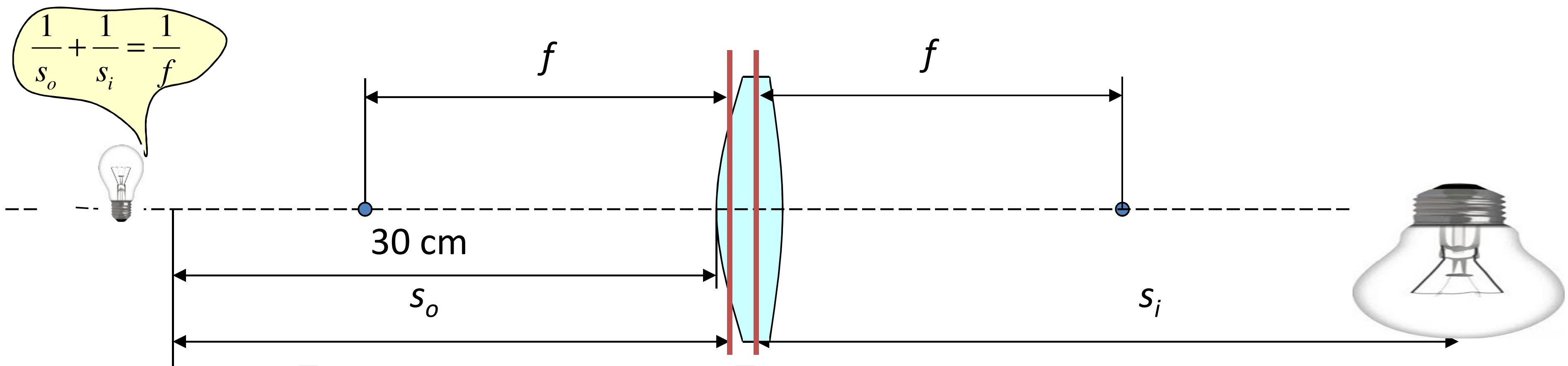
3. Calculate object distance, s_o, measured from principal plane
4. Calculate image distance:

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}$$

5. Calculate magnification, $m_T = -s_i/s_o$

Thick Lens: example

Find the image distance for an object positioned 30 cm from the vertex of a double convex lens having radii 20 cm and 40 cm, a thickness of 1 cm and n_l=1.5



$$\frac{1}{f}=(n_l-1)\left[\frac{1}{R_1}-\frac{1}{R_2}+\frac{(n_l-1)d_l}{n_l R_1 R_2}\right]=0.5\left[\frac{1}{20}-\frac{1}{-40}-\frac{0.5\cdot 1}{1.5\cdot 20\cdot 40}\right]\frac{1}{\text{cm}}$$

$$f=26.8\,\text{cm}$$

$$h_1=-\frac{26.8\cdot 0.5\cdot 1}{-40\cdot 1.5}\text{cm}=0.22\text{cm}$$

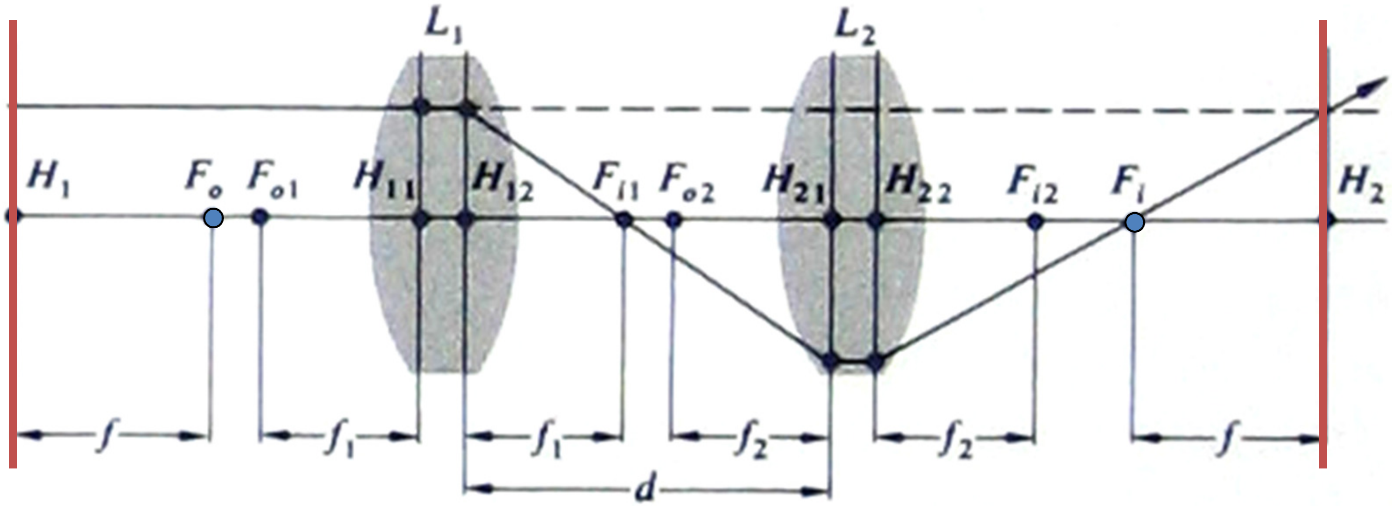
$$h_2=-\frac{26.8\cdot 0.5\cdot 1}{20\cdot 1.5}\text{cm}=-0.44\text{cm}$$

$$s_o=30\text{cm}+0.22\text{cm}=30.22\,\text{cm}$$

$$\frac{1}{30.22\text{cm}}+\frac{1}{s_i}=\frac{1}{26.8\text{cm}}$$

$$s_i=238\,\text{cm}$$

Compound Thick Lens



Can use two principal points (planes) and effective focal length *f* to describe propagation of rays through any compound system

Note: any ray passing through the first principal plane will emerge at the same height at the second principal plane

For 2 lenses (above):

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

$$\overline{H_{11}H_1} = fd/f_2$$

$$\overline{H_{22}H_2} = fd/f_1$$

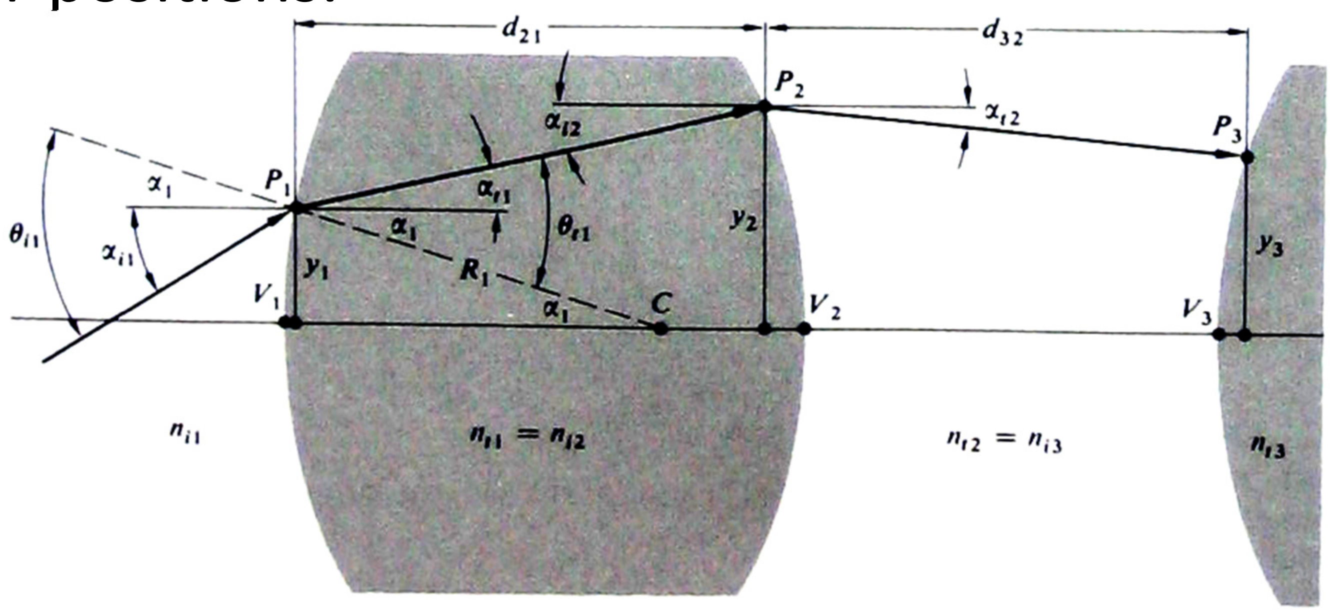
Example: page 246

Ray Tracing

- Even the thick lens equation makes approximations and assumptions
  - Spherical lens surfaces
  - Paraxial approximation
  - Alignment with optical axis
- The only physical concepts we applied were
  - Snell's law: $n_i \sin\theta_i = n_t \sin\theta_t$
  - Law of reflection: $\theta_t = \theta_i$ (in the case of mirrors)
- Can we do better? Can we solve for the paths of the rays exactly?
  - Sure, no problem! But it is a lot of work.
  - Computers are good at doing lots of work (without complaining)

Ray Tracing

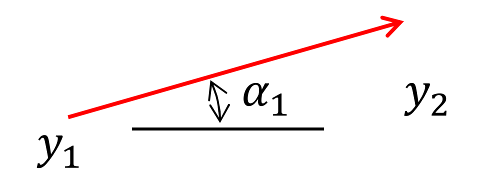
- We will still make the assumptions of
  - Paraxial rays
  - Lenses aligned along optical axis
- We will make no assumptions about the lens thickness or positions.
- Geometry:



Ray Tracing

- At a given point along the optical axis, each ray can be uniquely represented by two numbers:
  - Distance from optical axis, y_i
  - Angle with respect to optical axis, α_i
- If the ray does not encounter an optical element its distance from the optical axis changes according to the *transfer equation*:

$$y_2 = y_1 + d_1 \alpha_1$$



  - This assumes the paraxial approximation $\sin \alpha_1 \approx \alpha_1$

Ray Tracing

- At a given point along the optical axis, each ray can be uniquely represented by two numbers:
  - Distance from optical axis, y_i
  - Angle with respect to optical axis, α_i
- When the ray encounters a surface of a material with a different index of refraction, its angle will change according to the *refraction equation*:

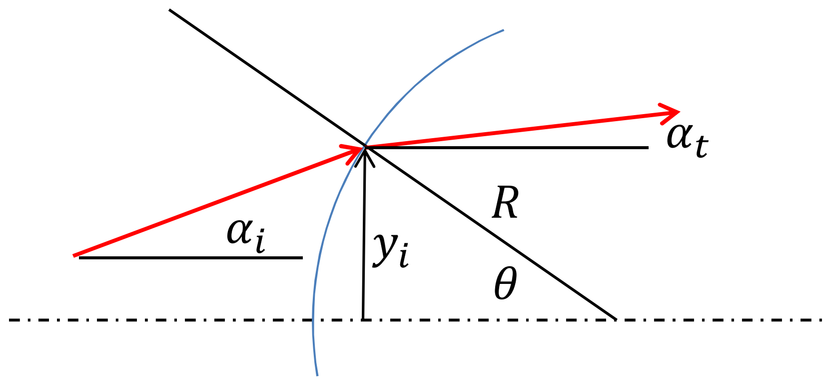
$$n_{t1}\alpha_{t1} = n_{i1}\alpha_{i1} - D_1 y_1$$

$$D_1 = \frac{n_{t1} - n_{i1}}{R_1}$$

  - Also assumes the paraxial approximation

Ray Tracing

- Geometry used for the refraction equation:



$$\sin\theta = \frac{y_i}{R} \approx \theta$$

$$\theta_i = \alpha_i + \theta = \alpha_i + y_i/R$$

$$\theta_t = \alpha_t + \theta = \alpha_t + y_i/R$$

$$n_i\theta_i = n_t\theta_t$$

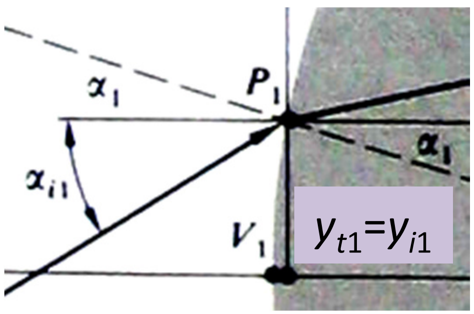
$$n_i\alpha_i + \frac{n_i y_i}{R} = n_t\alpha_t + \frac{n_t y_i}{R}$$

$$n_t\alpha_t = n_i\alpha_i - \left(\frac{n_t - n_i}{R}\right) y_i$$

Matrix Treatment: Refraction

At any point of space need 2 parameters to fully specify ray: distance from axis (y) and inclination angle (α) with respect to the optical axis. Optical element changes these ray parameters.

Refraction:



$$n_{t1}\alpha_{t1} = n_{i1}\alpha_{i1} - D_1 y_{i1}$$

$$y_{t1} = 0 \cdot n_{i1}\alpha_{i1} + y_{i1}$$

note: paraxial approximation

Reminder:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \alpha \\ y \end{pmatrix} \equiv \begin{pmatrix} A\alpha + By \\ C\alpha + Dy \end{pmatrix}$$

Equivalent matrix representation:

$$\begin{pmatrix} n_{t1}\alpha_{t1} \\ y_{t1} \end{pmatrix} = \begin{pmatrix} 1 & -D_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} n_{i1}\alpha_{i1} \\ y_{i1} \end{pmatrix}$$

$$\mathbf{r}_{t1} = \mathbf{R}_1 \mathbf{r}_{i1}$$

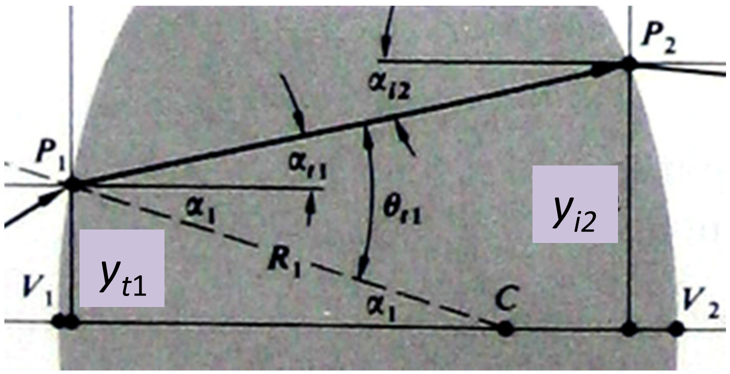
≡ $\mathbf{r}_{i1}$ - input ray

≡ $\mathbf{R}_1$ - ***refraction matrix***

≡ $\mathbf{r}_{t1}$ - output ray

Matrix: Transfer Through Space

Transfer:



$$n_{i2}\alpha_{i2} = n_{t1}\alpha_{t1} + 0 \cdot y_{t1}$$

$$y_{i2} = d_{21} \cdot \alpha_{t1} + y_{t1}$$

Equivalent matrix presentation:

$$\begin{pmatrix} n_{i2}\alpha_{i2} \\ y_{i2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d_{21}/n_{t1} & 1 \end{pmatrix} \begin{pmatrix} n_{t1}\alpha_{t1} \\ y_{t1} \end{pmatrix}$$

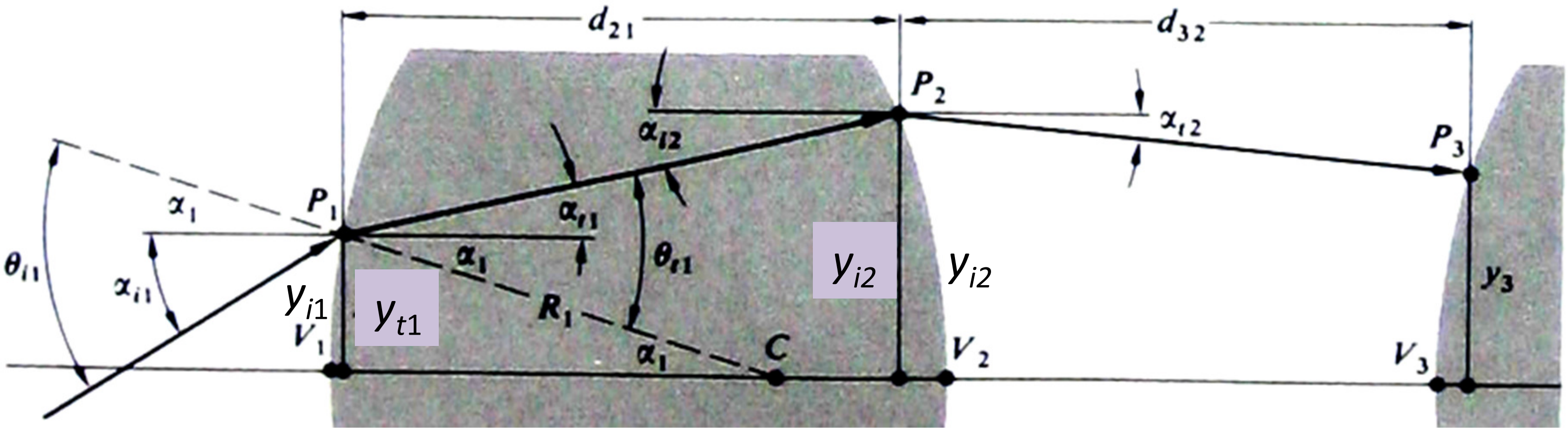
$\equiv \mathbf{r}_{t1}$ - input ray

$\equiv \mathbf{T}_{21}$ - ***transfer matrix***

$\equiv \mathbf{r}_{i2}$ - output ray

$$\mathbf{r}_{i2} = \mathbf{T}_{21}\mathbf{r}_{t1}$$

System Matrix



$\mathbf{r}_{i1} \rightarrow \mathbf{r}_{t1} \longrightarrow \mathbf{r}_{i2} \rightarrow \mathbf{r}_{t2} \longrightarrow \mathbf{r}_{i3} \rightarrow \mathbf{r}_{t3}$

$\mathbf{R}_1$ $\mathbf{T}_{21}$ $\mathbf{T}_{32}$ $\mathbf{R}_3$

$$\mathbf{r}_{t1} = \mathbf{R}_1 \mathbf{r}_{i1}$$

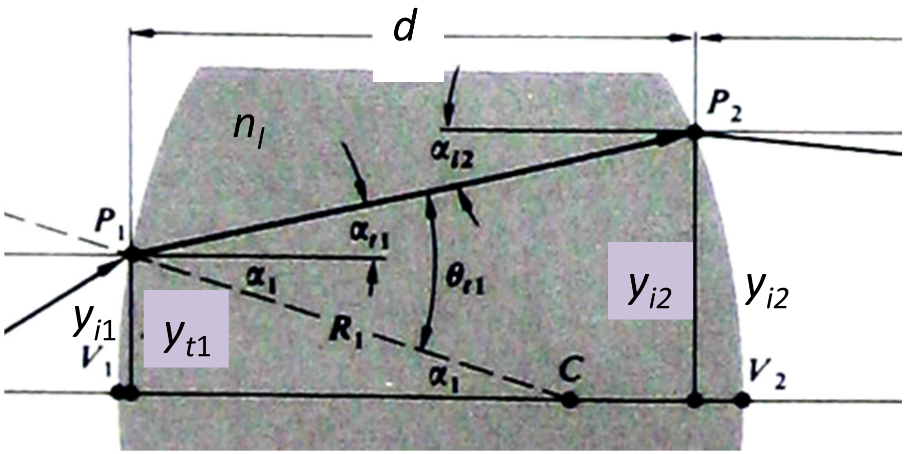
$$\mathbf{r}_{i2} = \mathbf{T}_{21} \mathbf{r}_{t1} = \mathbf{T}_{21} \mathbf{R}_1 \mathbf{r}_{i1}$$

Thick lens ray transfer: $$\mathbf{r}_{t2} = \mathbf{R}_2 \mathbf{T}_{21} \mathbf{R}_1 \mathbf{r}_{i1}$$

System matrix: $$\mathbf{A} = \mathbf{R}_2 \mathbf{T}_{21} \mathbf{R}_1 \rightarrow \mathbf{r}_{t2} = \mathbf{A}\ \mathbf{r}_{i1}$$

Can treat any system with single system matrix

Thick Lens Matrix



$$\mathbf{A} = \mathbf{R}_2\mathbf{T}_{21}\mathbf{R}_1$$

$$\mathbf{R} = \begin{pmatrix} 1 & -\text{D} \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ d/n & 1 \end{pmatrix}$$

Reminder:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} Aa+Bc & Ab+Bd \\ Ca+Dc & Cb+Dd \end{pmatrix}$$

$$\frac{1}{f} = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\mathbf{A} = \begin{pmatrix} 1 & -\text{D}_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ d_l/n_l & 1 \end{pmatrix}\begin{pmatrix} 1 & -\text{D}_1 \\ 0 & 1 \end{pmatrix}$$

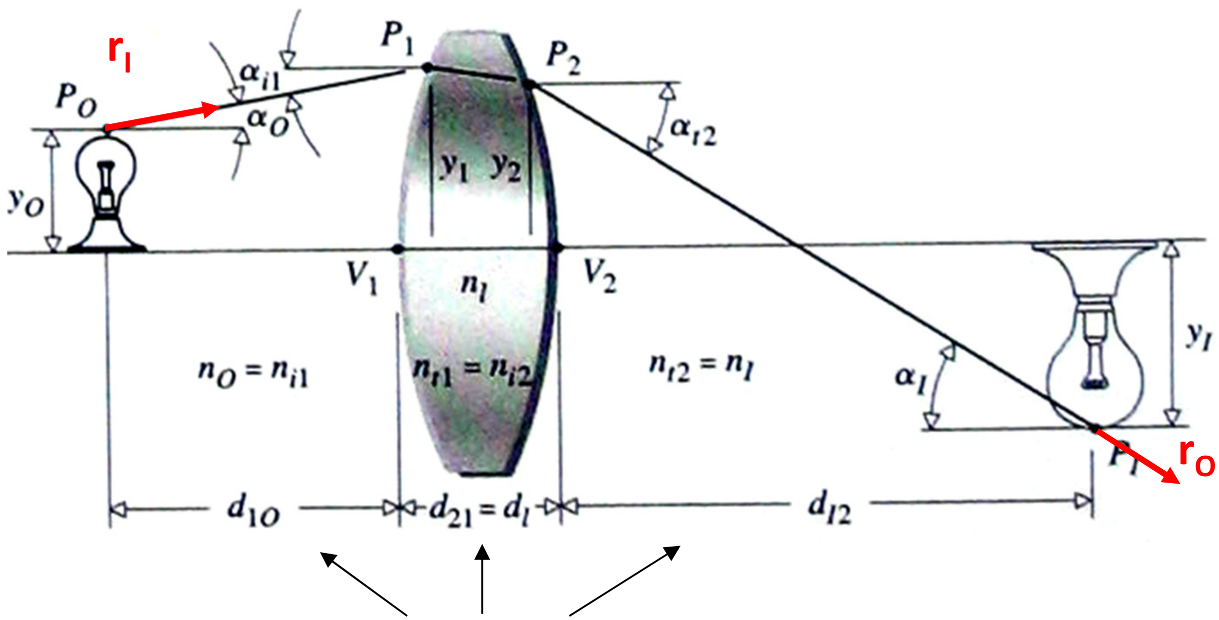
system matrix of thick lens

$$\mathbf{A} = \begin{pmatrix} 1 - \dfrac{\text{D}_2 d_l}{n_l} & -\text{D}_1 - \text{D}_2 - + \dfrac{\text{D}_1\text{D}_2 d_l}{n_l} \\ \dfrac{d_l}{n_l} & 1 - \dfrac{\text{D}_1 d_l}{n_l} \end{pmatrix}$$

For thin lens d_l=0

$$\mathbf{A} = \begin{pmatrix} 1 & -1/f \\ 0 & 1 \end{pmatrix}$$

Matrix Treatment: example



$$\mathbf{r}_I = \mathbf{T}_1 \mathbf{A}_l \mathbf{T}_2 \mathbf{r}_O$$

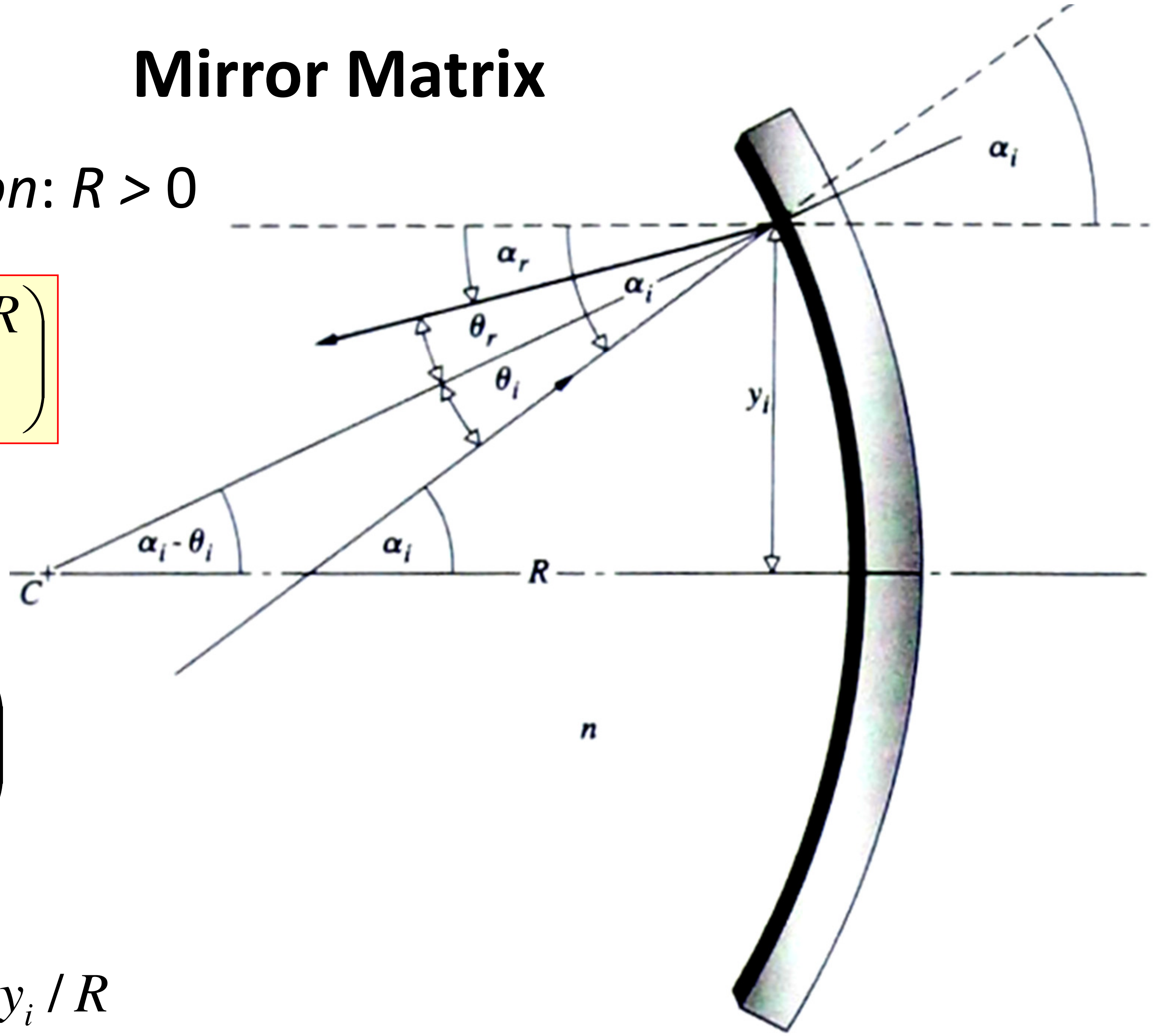
$$\begin{pmatrix} n_I \alpha_I \\ y_I \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d_{I2}/n_I & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d_{1O}/n_O & 1 \end{pmatrix} \begin{pmatrix} n_O \alpha_O \\ y_O \end{pmatrix}$$

(Detailed example with thick lenses and numbers: page 250)

Mirror Matrix

Sign convention: $R > 0$

$$\mathbf{M} = \begin{pmatrix} -1 & 2n/R \\ 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} n\alpha_r \\ y_r \end{pmatrix} = \mathbf{M} \begin{pmatrix} n\alpha_i \\ y_i \end{pmatrix}$$

$$y_r = y_i$$

$$n\alpha_r = -n\alpha_i + 2ny_i / R$$

$$\alpha_r = -\alpha_i + 2y_i / R$$

Tray Tracing Example

- Transfer matrix (distance d in medium n_1):

$$\begin{pmatrix} n_2\alpha_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d/n_1 & 1 \end{pmatrix} \begin{pmatrix} n_1\alpha_1 \\ y_1 \end{pmatrix}$$

- Refraction matrix (spherical surface)

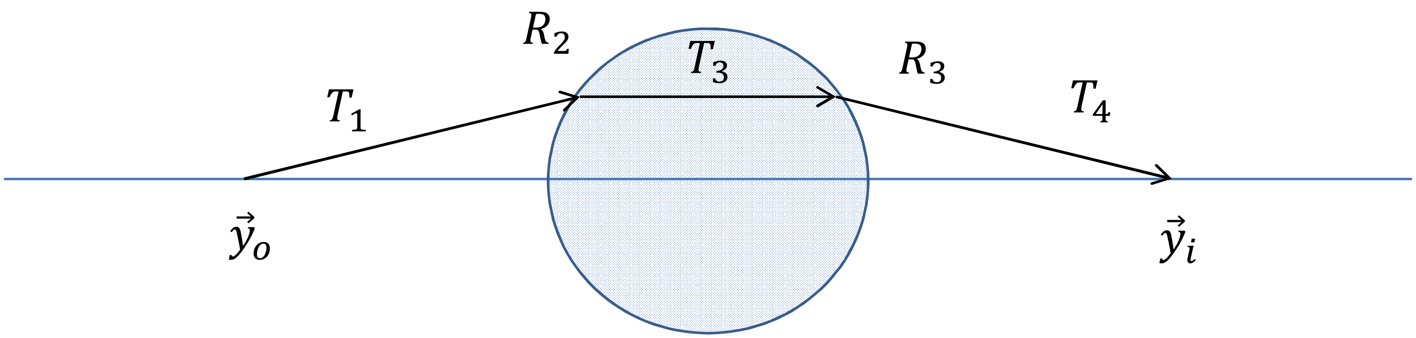
$$\begin{pmatrix} n_2\alpha_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -D \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n_1\alpha_1 \\ y_1 \end{pmatrix}$$

$$D = \frac{n_2 - n_1}{R}$$

- This example:

$$\vec{y}_o = T_5 R_4 T_3 R_2 T_1 \vec{y}_i$$

Ray Tracing Example



- Initial ray:

$$\vec{y}_o = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

- Final ray should cross the optical axis at a distance s_i from the second vertex.
- Multiply the matrices, solve for s_i…

Ray Tracing Example

- Use Mathematica…

In[10]:=
```
(*  Propagate ray to the image point *)
yi = tm[si, 1].y4
```

Out[10]=

$$\left\{\left\{\alpha + \frac{(1-n)\ so\alpha}{r} - \frac{(-1+n)\left(so\alpha + \frac{2\ r\left(\alpha + \frac{(1-n)\ so\alpha}{r}\right)}{n}\right)}{r}\right\},\left\{so\alpha + \frac{2\ r\left(\alpha + \frac{(1-n)\ so\alpha}{r}\right)}{n} + si\left(\alpha + \frac{(1-n)\ so\alpha}{r} - \frac{(-1+n)\left(so\alpha + \frac{2\ r\left(\alpha + \frac{(1-n)\ so\alpha}{r}\right)}{n}\right)}{r}\right)\right\}\right\}$$

In[13]:=
```
(*  The condition for an image is that the ray crosses the optical axis
 at the image point.  So we need to solve for si as a function of so.  *)
solution = Solve[yi[[2]] == 0, {si}]
```

Out[13]=

$$\left\{\left\{si \to \frac{r\ (2\ r + 2\ so - n\ so)}{-2\ r + n\ r - 2\ so + 2\ n\ so}\right\}\right\}$$

Ray Tracing Example

- Use Mathematica…

In[10]:=
```
(* Propagate ray to the image point *)
yi = tm[si, 1].y4
```

Out[10]=

$$\left\{\left\{\alpha+\frac{(1-n)\,so\,\alpha}{r}-\frac{(-1+n)\left(so\,\alpha+\frac{2\,r\left(\alpha+\frac{(1-n)\,so\,\alpha}{r}\right)}{n}\right)}{r}\right\},\left\{so\,\alpha+\frac{2\,r\left(\alpha+\frac{(1-n)\,so\,\alpha}{r}\right)}{n}+si\left(\alpha+\frac{(1-n)\,so\,\alpha}{r}-\frac{(-1+n)\left(so\,\alpha+\frac{2\,r\left(\alpha+\frac{(1-n)\,so\,\alpha}{r}\right)}{n}\right)}{r}\right)\right\}\right\}$$

In[13]:=
```
(* The condition for an image is that the ray crosses the optical axis
 at the image point. So we need to solve for si as a function of so. *)
solution = Solve[yi[[2]] == 0, {si}]
```

Out[13]=

$$\left\{\left\{si\to\frac{r\,(2\,r+2\,so-n\,so)}{-2\,r+n\,r-2\,so+2\,n\,so}\right\}\right\}$$

Ray Tracing Example

- Use Mathematica…
  - Object position was at the focal point of the first refracting surface:

$$s_o = \frac{R}{n-1}$$

```
In[15]:=  FullSimplify[si /. solution/. {so → r / (n - 1)}]

Out[15]=  {r/(-1 + n)}
```

- It works!