

Physics 42200
Waves & Oscillations

Lecture 24 – Review
 Spring 2016 Semester

Midterm Exam:

Date: Thursday, March 10th
 Time: 8:00 – 10:00 pm
 Room: MSEE B012
 Material: French, chapters 1-8

You can bring one double sided page
 of notes, formulas, examples, etc.

Review

1. Simple harmonic motion (one degree of freedom)
 - mass/spring, pendulum, floating objects, RLC circuits
 - damped harmonic motion
2. Forced harmonic oscillators
 - amplitude/phase of steady state oscillations
 - transient phenomena
3. Coupled harmonic oscillators
 - masses/springs, coupled pendula, RLC circuits
 - forced oscillations
4. Uniformly distributed discrete systems
 - masses on string fixed at both ends
 - lots of masses/springs

Review

5. Continuously distributed systems (standing waves)
 - string fixed at both ends
 - sound waves in pipes (open end/closed end)
 - transmission lines
 - Fourier analysis
6. ~~Progressive waves in continuous systems~~
 - ~~reflection/transmission coefficients~~

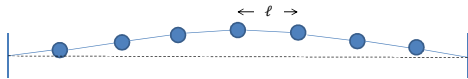
Uniformly Distributed Discrete Systems



Equations of motion for masses in the middle:

$$\ddot{x}_i + 2(\omega_0)^2 x_i - (\omega_0)^2 (x_{i-1} + x_{i+1}) = 0$$

$$(\omega_0)^2 = k/m$$



$$\ddot{y}_n + 2(\omega_0)^2 y_n - (\omega_0)^2 (y_{n+1} + y_{n-1}) = 0$$

$$(\omega_0)^2 = T/m\ell$$

Uniformly Distributed Discrete Masses

- Proposed solution:

$$x_n(t) = A_n \cos \omega t$$

$$\frac{A_{n-1} + A_{n+1}}{A_n} = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

- We solved this to determine A_n and ω_k :

$$A_{n,k} = C \sin\left(\frac{nk\pi}{N+1}\right)$$

Amplitude of mass n
oscillating in normal
mode k

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

Frequency of normal
mode k

- General solution:

$$x_n(t) = \sum_{k=1}^N a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \delta_k)$$

Vibrations of Continuous Systems

- General solution for mass n :

$$x_n(t) = \sum_{k=1}^N a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \delta_k)$$

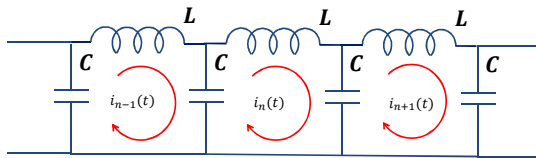
- Orthogonality relation:

$$\sum_{k=1}^N \sin\left(\frac{mk\pi}{N+1}\right) \sin\left(\frac{nk\pi}{N+1}\right) = \frac{N}{2} \delta_{mn}$$

- Solution to initial value problem:

$$\sum_{n=1}^N x_n(0) \sin\left(\frac{nk\pi}{N+1}\right) = \frac{N}{2} a_k \cos \delta_k$$

Lumped LC Circuit



$$-L \frac{di_n}{dt} - \frac{1}{C} \int (i_n - i_{n+1}) dt - \frac{1}{C} \int (i_n - i_{n-1}) dt = 0$$

$$\frac{d^2 i_n}{dt^2} + 2\omega_0^2 i_n - \omega_0^2 (i_{n-1} + i_{n+1}) = 0$$

This is the exact same problem as the previous two examples.

Forced Coupled Oscillators

- Qualitative features are the same:
 - Motion can be decoupled into a set of N independent oscillator equations (normal modes)
 - Amplitude of normal mode oscillations are large when driven with the frequency of the normal mode
 - Phase difference approaches $\pi/2$ at resonance
- You should be able to anticipate the qualitative behavior when coupled oscillators are driven by a periodic force.**

Continuous Distributions

Limit as $N \rightarrow \infty$ and $m/\ell \rightarrow \mu$:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

Boundary conditions specified at $x = 0$ and $x = L$:

- Fixed ends: $y(0) = y(L) = 0$
- Maximal motion at ends: $\dot{y}(0) = \dot{y}(L) = 0$
- Mixed boundary conditions

Normal modes will be of the form

$$y_n(x, t) = a_n \sin(k_n x) \cos(\omega_n t - \alpha_n)$$

or
$$y_n(x, t) = a_n \cos(k_n x) \cos(\omega_n t - \alpha_n)$$

Properties of the Solutions

$$y(L, t) \sim \sin k_n L = 0 \quad \Rightarrow \quad k_n L = n\pi$$

mode	wavelength	frequency
first	$2L$	$\frac{v}{2L}$
second	L	$\frac{v}{L}$
third	$\frac{2L}{3}$	$\frac{3v}{2L}$
fourth	$\frac{L}{2}$	$\frac{2v}{L}$

$$\lambda_n = \frac{2L}{n}$$

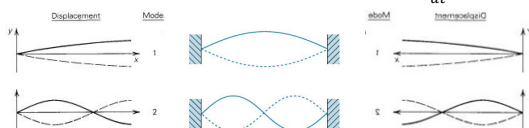
$$\omega_n = \frac{n\pi v}{L}$$

$$f_n = \frac{nv}{2L}$$

Boundary Conditions

• Examples:

- String fixed at both ends: $y(0) = y(L) = 0$
- Organ pipe open at one end: $\dot{y}(0) = \dot{y}(L) = 0$
 - Driving end has maximal pressure amplitude
- Organ pipe closed at one end: $\dot{y}(0) = 0, y(L) = 0$
- Transmission line open at one end: $i(L) = 0$
- Transmission line shorted at one end: $v(L) \propto \frac{di(L)}{dt} = 0$



Fourier Analysis

- Normal modes satisfying $y(0) = y(L) = 0$:

$$y_n(x, t) = a_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \alpha_n)$$

- General solution:

$$y(x, t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \alpha_n)$$

- Initial conditions:

$$y(x, 0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos(\alpha_n) = \sum_{n=1}^{\infty} a'_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\dot{y}(x, 0) = -\sum_{n=1}^{\infty} a_n \omega_n \sin\left(\frac{n\pi x}{L}\right) \sin(\alpha_n) = \sum_{n=1}^{\infty} b'_n \sin\left(\frac{n\pi x}{L}\right)$$

Fourier Analysis

- Fourier sine transform:

$$u(x) = \sum_{n=1}^{\infty} a'_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a'_n = \frac{2}{L} \int_0^L u(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

- Fourier cosine transform:

$$b'_n = \frac{2}{L} \int_0^L v(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

Fourier Analysis

$$a'_n = a_n \cos \alpha_n$$

$$b'_n = a_n \omega_n \sin \alpha_n$$

Solve for amplitudes:

$$a_n = \sqrt{a'^2_n + \frac{b'^2_n}{\omega_n^2}}$$

Solve for phase:

$$\tan \alpha_n = \frac{b'_n}{a'_n \omega_n}$$

Fourier Analysis

- **Suggestion: don't simply rely on these formulas – use your knowledge of the boundary conditions and initial conditions.**

- Example:

- If you are given $\dot{y}(x, 0) = 0$ and $y(0) = y(L) = 0$ then you know that solutions are of the form

$$y(x, t) = \sum a_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

- If you are given $y(x, 0) = 0$ and $\dot{y}(0) = \dot{y}(L) = 0$ then solutions are of the form

$$y(x, t) = \sum_{\text{odd } n} a_n \sin\left(\frac{n\pi x}{L}\right) \sin \omega_n t$$

Progressive Waves

- Far from the boundaries, other descriptions are more transparent:
 $y(x, t) = f(x \pm vt)$

- The Fourier transform gives the frequency components:

$$\begin{aligned} A(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cos(kx) dx \\ g(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \cos(kx) dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(k) \sin(kx) dk \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \sin(kx) dx \end{aligned}$$

- Narrow pulse in space \rightarrow wide range of frequencies
- Pulse spread out in space \rightarrow narrow range of frequencies

Properties of Progressive Waves

- Power carried by a wave:

- String with tension T and mass per unit length μ

$$P = \frac{1}{2} \mu \omega^2 A^2 v = \frac{1}{2} Z \omega^2 A^2$$

- Impedance of the medium:

$$Z = \mu v = T/v$$

- Important properties:

- **Impedance is a property of the medium, not the wave**
- **Energy and power are proportional to the square of the amplitude**

Reflections

- Wave energy is reflected by discontinuities in the impedance of a system
- Reflection and transmission coefficients:
 - The wave is incident and reflected in medium 1
 - The wave is transmitted into medium 2

$$\rho = \frac{Z_2 - Z_1}{Z_1 + Z_2}$$

$$\tau = \frac{2Z_2}{Z_1 + Z_2}$$

Important: when is this negative?
Always positive

- Wave amplitudes:

$$A_r = \rho A_i$$

$$A_t = \tau A_i$$

Reflected and Transmitted Power

- Power is proportional to the square of the amplitude.
 - Reflected power: $P_r = \rho^2 P_i$
 - Transmitted power: $P_t = \tau^2 P_i$
- You should be able to demonstrate that energy is conserved:**
ie, show that $P_i = P_r + P_t$

Examples from Previous Midterms

3. Consider a string with tension T on which N beads with equal mass m and spacing ℓ are attached. If all the beads are motionless at $t = 0$ then the motion of an arbitrary bead k can be described by a sum over normal modes of oscillation:

$$y_k(t) = \sum_{n=1}^N a_n \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_n t)$$

where

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right).$$

Find an expression for a_n that will solve the initial value problem where all masses are initially in their equilibrium position except for bead m , which is displaced by an amplitude A at $t = 0$.

It may be useful to recall that

$$\sum_{k=1}^N \sin\left(\frac{nk\pi}{N+1}\right) \sin\left(\frac{mk\pi}{N+1}\right) = \frac{N}{2} \delta_{nm}$$

$$\begin{aligned}
 3. \quad y_k(t) &= \sum_{n=1}^N a_n \sin\left(\frac{n k \pi}{N+1}\right) \cos \omega_n t \\
 y_k(0) &= \sum_{n=1}^N a_n \sin\left(\frac{n k \pi}{N+1}\right) = A \delta_{mk} \\
 \left. \sum_{k=1}^N \sum_{n=1}^N a_n \sin\left(\frac{n k \pi}{N+1}\right) \sin\left(\frac{m k \pi}{N+1}\right) \right\} &= \frac{N}{2} \delta_{nm} \\
 &= \sum_{k=1}^N y_k(0) \sin\left(\frac{m k \pi}{N+1}\right) \\
 &= \frac{N}{2} \sum_{k=1}^N a_n \delta_{nm} = \frac{N}{2} a_m \\
 \text{So } a_n &= \frac{2}{N} \sum_{k=1}^N y_k(0) \sin\left(\frac{n k \pi}{N+1}\right) \\
 &= \frac{2}{N} \sum_{k=1}^N A \delta_{mk} \sin\left(\frac{n k \pi}{N+1}\right) \\
 &= \frac{2A}{N} \sin\left(\frac{m n \pi}{N+1}\right)
 \end{aligned}$$

Examples from Previous Midterms

- Lots of other examples from previous assignments linked from <http://www.physics.purdue.edu/~mjones/>
- In particular:
 - [Spring, 2015](#)
 - [Spring, 2014](#)
 - [Spring, 2013](#)

That's all for now...

- Study these topics – make sure you understand the examples and assignment questions.
- Midterm exams from previous years are also available on the web.
- Next topics: **waves applied to optics.**
