PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 24 – Review

Spring 2016 Semester

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Midterm Exam:

Date:	Thursday, March 10th
Time:	8:00 – 10:00 pm
Room:	MSEE B012
Material:	French, chapters 1-8

You can bring one double sided page of notes, formulas, examples, etc.

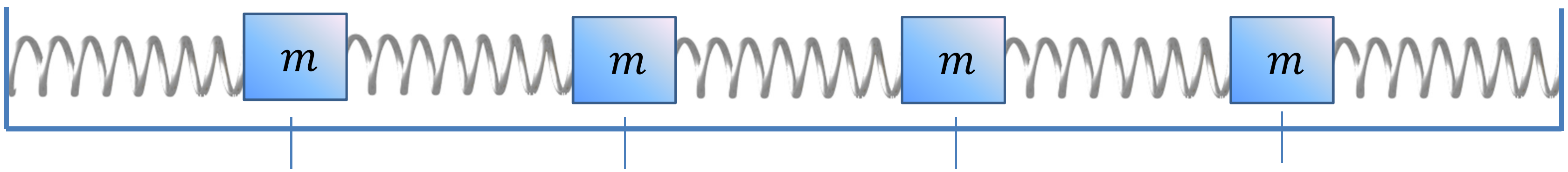
Review

1. Simple harmonic motion (one degree of freedom)
   - mass/spring, pendulum, floating objects, RLC circuits
   - damped harmonic motion
2. Forced harmonic oscillators
   - amplitude/phase of steady state oscillations
   - transient phenomena
3. Coupled harmonic oscillators
   - masses/springs, coupled pendula, RLC circuits
   - forced oscillations
4. Uniformly distributed discrete systems
   - masses on string fixed at both ends
   - lots of masses/springs

Review

5. Continuously distributed systems (standing waves)
   - string fixed at both ends
   - sound waves in pipes (open end/closed end)
   - transmission lines
   - Fourier analysis
6. Progressive waves in continuous systems
   - reflection/transmission coefficients

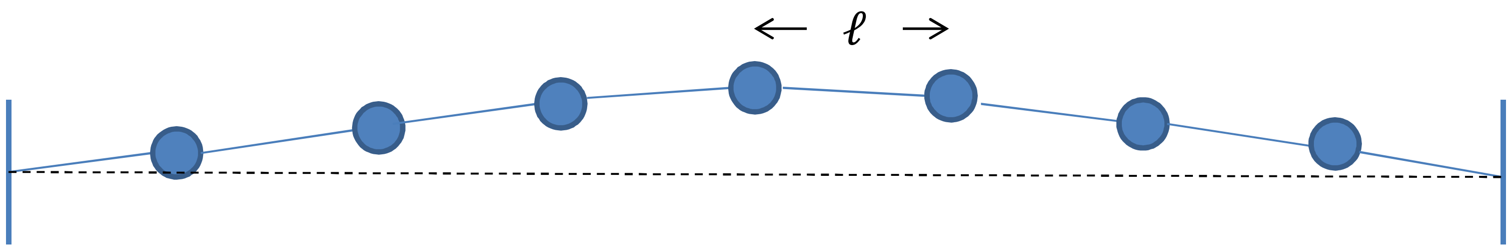
Uniformly Distributed Discrete Systems



Equations of motion for masses in the middle:

$$\ddot{x}_i + 2(\omega_0)^2 x_i - (\omega_0)^2(x_{i-1} + x_{i+1}) = 0$$

$$(\omega_0)^2 = k/m$$



$$\ddot{y}_n + 2(\omega_0)^2 y_n - (\omega_0)^2(y_{n+1} + y_{n-1}) = 0$$

$$(\omega_0)^2 = T/m\ell$$

Uniformly Distributed Discrete Masses

- Proposed solution:

$$x_n(t) = A_n \cos \omega t$$

$$\frac{A_{n-1} + A_{n+1}}{A_n} = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

- We solved this to determine A_n and ω_k:

$$A_{n,k} = C \sin\left(\frac{nk\pi}{N+1}\right)$$

Amplitude of mass n oscillating in normal mode k

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

Frequency of normal mode k

- General solution:

$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \delta_k)$$

Vibrations of Continuous Systems

- General solution for mass n:

$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \delta_k)$$

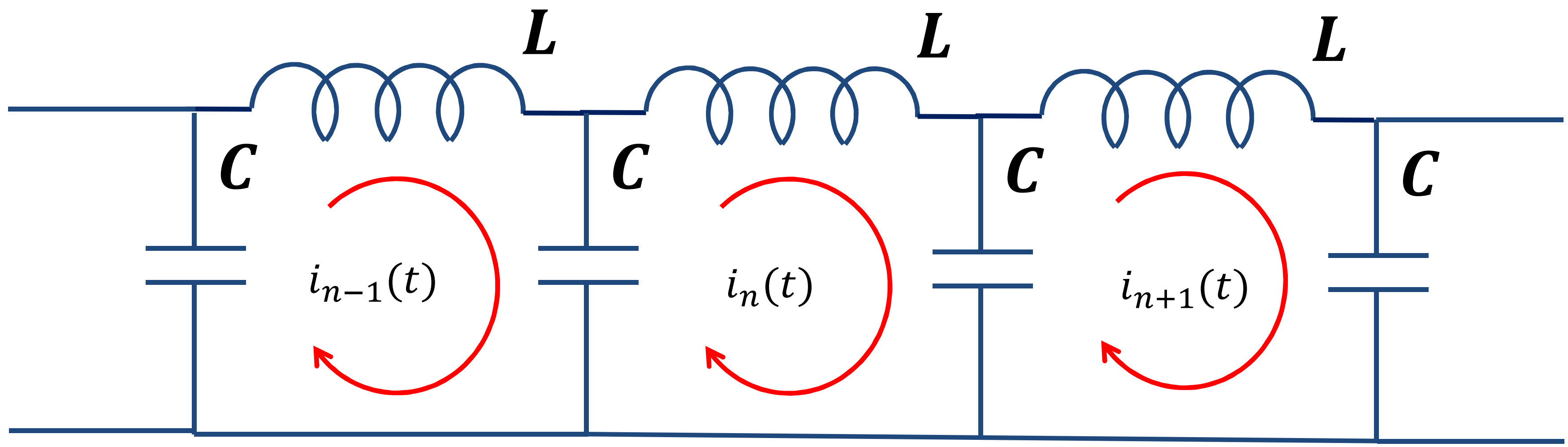
- Orthogonality relation:

$$\sum_{k=1}^{N} \sin\left(\frac{mk\pi}{N+1}\right) \sin\left(\frac{nk\pi}{N+1}\right) = \frac{N}{2}\ \delta_{mn}$$

- Solution to initial value problem:

$$\sum_{n=1}^{N} x_n(0) \sin\left(\frac{nk\pi}{N+1}\right) = \frac{N}{2} a_k\, cos\ \delta_k$$

Lumped LC Circuit



$$-L\frac{di_n}{dt} - \frac{1}{C}\int (i_n - i_{n+1})dt \; - \frac{1}{C}\int (i_n - i_{n-1})dt = 0$$

$$\frac{d^2 i_n}{dt^2} + 2\omega_0^2 i_n - \omega_0^2 (i_{n-1} + i_{n+1}) = 0$$

This is the exact same problem as the previous two examples.

Forced Coupled Oscillators

- Qualitative features are the same:
  - Motion can be decoupled into a set of N independent oscillator equations (normal modes)
  - Amplitude of normal mode oscillations are large when driven with the frequency of the normal mode
  - Phase difference approaches $\pi/2$ at resonance
- ***You should be able to anticipate the qualitative behavior when coupled oscillators are driven by a periodic force.***

Continuous Distributions

Limit as $N \to \infty$ and $m/\ell \to \mu$:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Boundary conditions specified at $x = 0$ and $x = L$:

- Fixed ends: $y(0) = y(L) = 0$
- Maximal motion at ends: $\dot{y}(0) = \dot{y}(L) = 0$
- Mixed boundary conditions

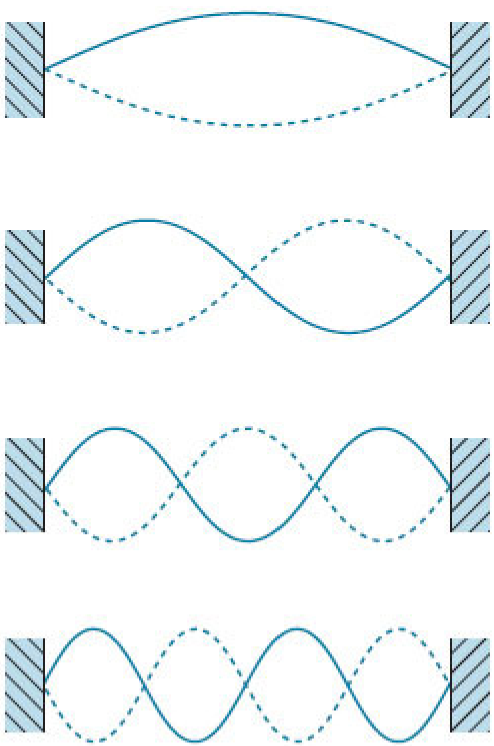
Normal modes will be of the form

$$y_n(x,t) = a_n \sin(k_n x)\cos(\omega_n t - \alpha_n)$$

or $$y_n(x,t) = a_n \cos(k_n x)\cos(\omega_n t - \alpha_n)$$

Properties of the Solutions

$$y(L,t) \sim \sin k_n L = 0 \quad \Rightarrow \quad k_n L = n\pi$$



mode	wavelength	frequency
first	$2L$	$\frac{v}{2L}$
second	L	$\frac{v}{L}$
third	$\frac{2L}{3}$	$\frac{3v}{2L}$
fourth	$\frac{L}{2}$	$\frac{2v}{L}$

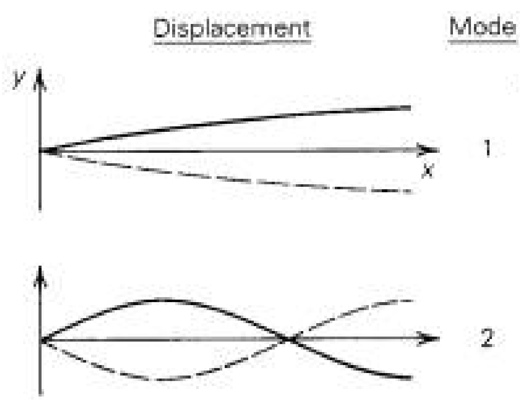
$$\lambda_n = \frac{2L}{n}$$

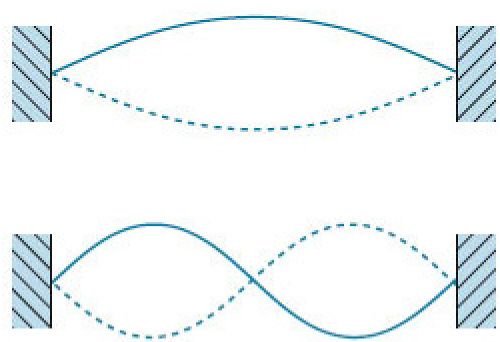
$$\omega_n = \frac{n\pi v}{L}$$

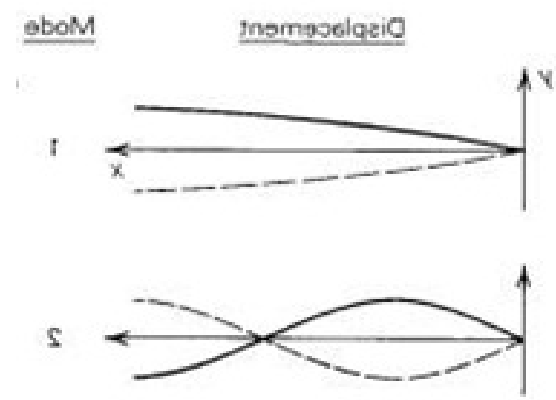
$$f_n = \frac{nv}{2L}$$

Boundary Conditions

- Examples:
  - String fixed at both ends: $y(0) = y(L) = 0$
  - Organ pipe open at one end: $\dot{y}(0) = \dot{y}(L) = 0$
    - Driving end has maximal pressure amplitude
  - Organ pipe closed at one end: $\dot{y}(0) = 0, y(L) = 0$
  - Transmission line open at one end: $i(L) = 0$
  - Transmission line shorted at one end: $v(L) \propto \frac{di(L)}{dt} = 0$







Fourier Analysis

- Normal modes satisfying $y(0) = y(L) = 0$:

$$y_n(x,t) = a_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \alpha_n)$$

- General solution:

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \alpha_n)$$

- Initial conditions:

$$y(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos(\alpha_n) = \sum_{n=1}^{\infty} a'_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\dot{y}(x,0) = -\sum_{n=1}^{\infty} a_n \omega_n \sin\left(\frac{n\pi x}{L}\right)\sin(\alpha_n) = \sum_{n=1}^{\infty} b'_n \sin\left(\frac{n\pi x}{L}\right)$$

Fourier Analysis

- Fourier sine transform:

$$u(x) = \sum_{n=1}^{\infty} a'_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a'_n = \frac{2}{L}\int_0^L u(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

- Fourier cosine transform:

$$b'_n = \frac{2}{L}\int_0^L v(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

Fourier Analysis

$$a'_n = a_n \cos \alpha_n$$
$$b'_n = a_n \omega_n \sin \alpha_n$$

Solve for amplitudes:

$$a_n = \sqrt{a'^2_n + \frac{b'^2_n}{\omega^2_n}}$$

Solve for phase:

$$\tan \alpha_n = \frac{b'_n}{a'_n \omega_n}$$

Fourier Analysis

- ***Suggestion: don't simply rely on these formulas – use your knowledge of the boundary conditions and initial conditions.***
- Example:
  - If you are given $\dot{y}(x,0) = 0$ and $y(0) = y(L) = 0$ then you know that solutions are of the form

$$y(x,t) = \sum a_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

  - If you are given $y(x,0) = 0$ and $y(0) = y(L) = 0$ then solutions are of the form

$$y(x,t) = \sum_{odd\ n} a_n \sin\left(\frac{n\pi x}{L}\right) \sin \omega_n t$$

Progressive Waves

- Far from the boundaries, other descriptions are more transparent:

$$y(x,t) = f(x \pm vt)$$

- The Fourier transform gives the frequency components:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cos(kx)\, dx$$

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \cos(kx)\, dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(k) \sin(kx)\, dk\ B(k)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \sin(kx)\, dx$$

- Narrow pulse in space ➔ wide range of frequencies
- Pulse spread out in space ➔ narrow range of frequencies

Properties of Progressive Waves

- Power carried by a wave:
  - String with tension T and mass per unit length μ

$$P = \frac{1}{2}\mu\omega^2 A^2 v = \frac{1}{2} Z\omega^2 A^2$$

- Impedance of the medium:

$$Z = \mu v = T/v$$

- Important properties:
  - ***Impedance is a property of the medium, not the wave***
  - ***Energy and power are proportional to the square of the amplitude***

Reflections

- Wave energy is reflected by discontinuities in the impedance of a system
- Reflection and transmission coefficients:
  - The wave is incident and reflected in medium 1
  - The wave is transmitted into medium 2

$$\boldsymbol{\rho = \frac{Z_2 - Z_1}{Z_1 + Z_2}}$$

Important: when is this negative?

$$\boldsymbol{\tau = \frac{2Z_2}{Z_1 + Z_2}}$$

Always positive

- Wave amplitudes:

$$A_r = \rho A_i$$
$$A_t = \tau A_i$$

Reflected and Transmitted Power

- Power is proportional to the square of the amplitude.
  - Reflected power: $P_r = \rho^2 P_i$
  - Transmitted power: $P_t = \tau^2 P_i$
- ***You should be able to demonstrate that energy is conserved:***

  ie, show that $\boldsymbol{P_i = P_r + P_t}$

Examples from Previous Midterms

3. Consider a string with tension T on which N beads with equal mass m and spacing ℓ are attached. If all the beads are motionless at $t = 0$ then the motion of an arbitrary bead k can be described by a sum over normal modes of oscillation:

$$y_k(t) = \sum_{n=1}^{N} a_n \sin\left(\frac{nk\pi}{N+1}\right)\cos(\omega_n t)$$

where

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right).$$

Find an expression for a_n that will solve the initial value problem where all masses are initially in their equilibrium position except for bead m, which is displaced by an amplitude A at $t = 0$.
It may be useful to recall that

$$\sum_{k=1}^{N} \sin\left(\frac{nk\pi}{N+1}\right)\sin\left(\frac{mk\pi}{N+1}\right) = \frac{N}{2}\delta_{nm}$$

3. $$y_k(t) = \sum_{n=1}^{N} a_n \sin\left(\frac{nk\pi}{N+1}\right) \cos \omega_n t$$

$$y_k(0) = \sum_{n=1}^{N} a_n \sin\left(\frac{nk\pi}{N+1}\right) = A\delta_{mk}$$

$$\sum_{k=1}^{N} \sum_{n=1}^{N} a_n \sin\left(\frac{nk\pi}{N+1}\right) \sin\left(\frac{mk\pi}{N+1}\right)$$

$$= \sum_{k=1}^{N} y_k(0) \sin\left(\frac{mk\pi}{N+1}\right)$$

$$= \frac{N}{2} \sum_{n=1}^{N} a_n \delta_{nm} = \frac{N}{2} a_m$$

$$\frac{N}{2}\delta_{nm}$$

So $$a_n = \frac{2}{N} \sum_{k=1}^{N} y_k(0) \sin\left(\frac{nk\pi}{N+1}\right)$$

$$= \frac{2}{N} \sum_{k=1}^{N} A\delta_{mk} \sin\left(\frac{nk\pi}{N+1}\right)$$

$$= \frac{2A}{N} \sin\left(\frac{mn\pi}{N+1}\right)$$

Examples from Previous Midterms

- Lots of other examples from previous assignments linked from

  http://www.physics.purdue.edu/~mjones/

- In particular:
  - Spring, 2015
  - Spring, 2014
  - Spring, 2013

That's all for now…

- Study these topics – make sure you understand the examples and assignment questions.

- Midterm exams from previous years are also available on the web.

- Next topics: ***waves applied to optics.***