

Physics 42200
Waves & Oscillations

Lecture 23 – Review
Spring 2016 Semester

Midterm Exam:

Date: Thursday, March 10th
Time: 8:00 – 10:00 pm
Room: MSEE B012
Material: French, chapters 1-8

You can bring one double sided page
of notes, formulas, examples, etc.

Review

1. Simple harmonic motion (one degree of freedom)
 - mass/spring, pendulum, floating objects, RLC circuits
 - damped harmonic motion
2. Forced harmonic oscillators
 - amplitude/phase of steady state oscillations
 - transient phenomena
3. Coupled harmonic oscillators
 - masses/springs, coupled pendula, RLC circuits
 - forced oscillations
4. Uniformly distributed discrete systems
 - masses on string fixed at both ends
 - lots of masses/springs

Review

5. Continuously distributed systems (standing waves)
 - string fixed at both ends
 - sound waves in pipes (open end/closed end)
 - transmission lines
 - Fourier analysis
6. ~~Progressive waves in continuous systems~~
 - ~~reflection/transmission coefficients~~

Simple Harmonic Motion

- Any system in which the force is opposite the displacement will oscillate about a point of stable equilibrium
- If the force is proportional to the displacement it will undergo simple harmonic motion
- Examples:
 - Mass/massless spring
 - Elastic rod (characterized by Young's modulus)
 - Floating objects
 - Torsion pendulum (shear modulus)
 - Simple pendulum
 - Physical pendulum
 - LC circuit

Simple Harmonic Motion

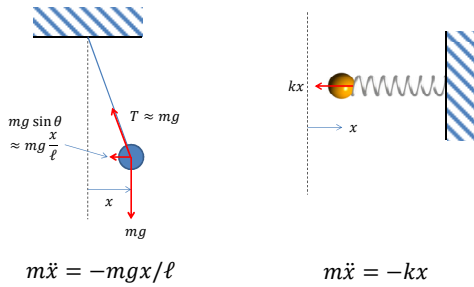
- You should be able to draw a free-body diagram and express the force in terms of the displacement.
- Use Newton's law: $m\ddot{x} = F$ or $I\ddot{\theta} = N$
- Write it in standard form:

$$\ddot{x} + \omega_0^2 x = 0$$
- Solutions are of the form:

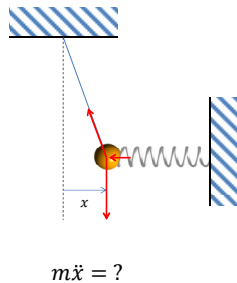
$$x(t) = A \cos(\omega_0 t - \delta)$$

$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t$$
- **You must be able to use the initial conditions to solve for the constants of integration**

Examples



Examples



Damped Harmonic Motion

- Damping forces remove energy from the system
- We will only consider cases where the force is proportional to the velocity: $F = -bv$
- You should be able to construct a free-body diagram and write the resulting equation of motion:

$$m\ddot{x} + b\dot{x} + kx = 0$$
 - You should be able to write it in the standard form:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$
- **You must be able to solve this differential equation!**

Damped Harmonic Motion

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$$

$$\text{Let } x(t) = Ae^{\alpha t}$$

- Characteristic polynomial:

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0$$

- Roots (use the quadratic formula):

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}$$

- Classification of solutions:

- Over-damped: $\gamma^2/4 - (\omega_0)^2 > 0$ (distinct real roots)
- Critically damped: $\gamma^2/4 = (\omega_0)^2$ (one root)
- Under-damped: $\gamma^2/4 - (\omega_0)^2 < 0$ (complex roots)

Damped Harmonic Motion

- Over-damped motion: $\gamma^2/4 - (\omega_0)^2 > 0$

$$x(t) = Ae^{-\frac{\gamma}{2}t} e^{t\sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}} + Be^{-\frac{\gamma}{2}t} e^{-t\sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}}$$

- Under-damped motion: $\gamma^2/4 - (\omega_0)^2 < 0$

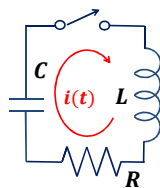
$$x(t) = Ae^{-\frac{\gamma}{2}t} e^{it\sqrt{(\omega_0)^2 - \frac{\gamma^2}{4}}} + Be^{-\frac{\gamma}{2}t} e^{-it\sqrt{(\omega_0)^2 - \frac{\gamma^2}{4}}}$$

- Critically damped motion:

$$x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$$

- **You must be able to use the initial conditions to solve for the constants of integration**

Example



Sum of potential differences:

$$-L \frac{di}{dt} - i(t)R - \frac{1}{C} \left(Q_0 + \int_0^t i(t) dt \right) = 0$$

Initial charge, Q_0 , defines the initial conditions.

Example

$$L \frac{di}{dt} + i(t)R + \frac{1}{C} \left(Q_0 + \int_0^t i(t) dt \right) = 0$$

Differentiate once with respect to time:

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i(t) = 0$$

$$\frac{d^2 i}{dt^2} + \gamma \frac{di}{dt} + \omega_0^2 i(t) = 0$$

Remember, the solution is $i(t)$ but the initial conditions might be in terms of $Q(t) = Q_0 + \int i(t) dt$

(See examples from the lecture notes...)

Forced Harmonic Motion

- Now the differential equation is

$$m\ddot{x} + b\dot{x} + kx = F(\omega) = F_0 \cos \omega t$$
- Driving function is not always given in terms of a real force... (think about non-inertial reference frames):

$$\ddot{y} + \gamma \dot{y} + \omega_0^2 y = -\frac{d^2 \eta}{dt^2} = C \omega^2 \cos \omega t$$

- General properties:
 - Steady state properties: $t \gg 1/\gamma$
 - Solution is $y(t) = A \cos(\omega t - \delta)$
 - Amplitude, A , and phase, δ , depend on ω

Forced Harmonic Motion

"Q" quantifies the amount of damping:

$$Q = \frac{\omega_0}{\gamma}$$

(large Q means small damping force)

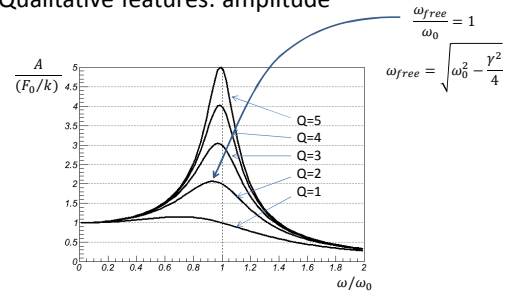
$$A(\omega) = \frac{F_0}{k} \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0} \right)^2 + \frac{1}{Q^2} \right]^{1/2}}$$

$$\delta = \tan^{-1} \left(\frac{1/Q}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}} \right)$$

But watch out when $F_0 = C \omega^2$

Resonance

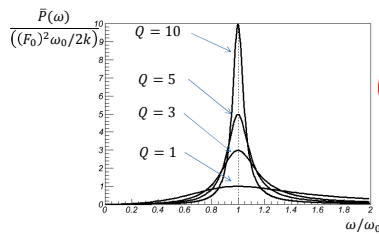
- Qualitative features: amplitude



Average Power

- The rate at which the oscillator absorbs energy is:

$$\bar{P}(\omega) = \frac{(F_0)^2 \omega_0}{2kQ} \frac{1}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}$$



Resonance

- Qualitative features: phase shift

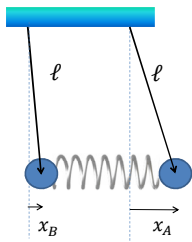
$$\delta = \tan^{-1} \left(\frac{1/Q}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}} \right)$$

$\delta \rightarrow 0$ at low frequencies

$\delta \rightarrow \pi$ at high frequencies

$\delta = \frac{\pi}{2}$ when $\omega = \omega_0$

Coupled Oscillators



- Restoring force on pendulum A:
 $F_A = -k(x_A - x_B)$
- Restoring force on pendulum B:
 $F_B = k(x_A - x_B)$

$$m\ddot{x}_A + \frac{mg}{\ell}x_A + k(x_A - x_B) = 0$$

$$m\ddot{x}_B + \frac{mg}{\ell}x_B - k(x_A - x_B) = 0$$

Coupled Oscillators

- You must be able to draw the free-body diagram and set up the system of equations.**

$$m\ddot{x}_A + \frac{mg}{\ell}x_A + k(x_A - x_B) = 0$$

$$m\ddot{x}_B + \frac{mg}{\ell}x_B - k(x_A - x_B) = 0$$

- You must be able to write this system as a matrix equation.**

$$\begin{pmatrix} \ddot{x}_A \\ \ddot{x}_B \end{pmatrix} + \begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 \end{pmatrix} \begin{pmatrix} x_A(t) \\ x_B(t) \end{pmatrix} = 0$$

Coupled Oscillators

- Assume solutions are of the form

$$\begin{pmatrix} x_A(t) \\ x_B(t) \end{pmatrix} = \begin{pmatrix} x_A \\ x_B \end{pmatrix} \cos(\omega t - \delta)$$

- Then,

$$\begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 - \omega^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 - \omega^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = 0$$

- You must be able to calculate the eigenvalues of a 2x2 or 3x3 matrix.**

- Calculate the determinant
- Calculate the roots by factoring the determinant or using the quadratic formula.

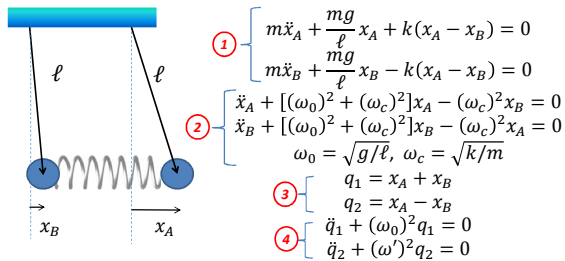
- These are the frequencies of the normal modes of oscillation.

Coupled Oscillators

- You must be able to calculate the eigenvectors of a 2x2 or 3x3 matrix
- General solution:
 $\vec{x}(t) = \mathbf{A}\vec{x}_1 \cos(\omega_1 t - \alpha) + \mathbf{B}\vec{x}_2 \cos(\omega_2 t - \beta) + \dots$
- You must be able to solve for the constants of integration using the initial conditions.

Coupled Discrete Systems

- The general method of calculating eigenvalues will always work, but for simple systems you should be able to decouple the equations by a change of variables.



Forced Oscillations

- We mainly considered the qualitative aspects
 - We did not analyze the behavior when damping forces were significant
- Main features:
 - Resonance occurs at each normal mode frequency
 - Phase difference is $\delta = \pi/2$ at resonance
- Example: x_A driven by the force $F(\omega) = F_0 \cos \omega t$
 - Calculate force term applied to normal coordinates
 $F_1(\omega) = F_2(\omega) = F_0 \cos \omega t$
 - Reduced to two one-dimensional forced oscillators:

$$\ddot{q}_1 + (\omega_0)^2q_1 = F_0/m \cos \omega t$$

$$\ddot{q}_2 + (\omega')^2q_2 = F_0/m \cos \omega t$$

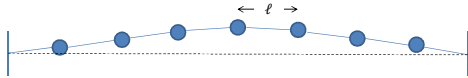
Uniformly Distributed Discrete Systems



Equations of motion for masses in the middle:

$$\ddot{x}_i + 2(\omega_0)^2 x_i - (\omega_0)^2 (x_{i-1} + x_{i+1}) = 0$$

$$(\omega_0)^2 = k/m$$



$$\ddot{y}_n + 2(\omega_0)^2 y_n - (\omega_0)^2 (y_{n+1} + y_{n-1}) = 0$$

$$(\omega_0)^2 = T/m\ell$$

Uniformly Distributed Discrete Masses

- Proposed solution:

$$\frac{x_n(t)}{A_n} = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2} \cos \omega t$$

- We solved this to determine A_n and ω_k :

$$A_{n,k} = C \sin\left(\frac{nk\pi}{N+1}\right)$$

Amplitude of mass n
oscillating in normal
mode k

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

Frequency of normal
mode k

- General solution:

$$x_n(t) = \sum_{k=1}^N a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \delta_k)$$

Vibrations of Continuous Systems

- General solution for mass n :

$$x_n(t) = \sum_{k=1}^N a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \delta_k)$$

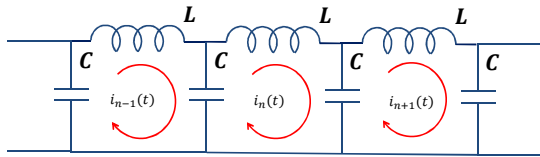
- Orthogonality relation:

$$\sum_{k=1}^N \sin\left(\frac{mk\pi}{N+1}\right) \sin\left(\frac{nk\pi}{N+1}\right) = \frac{N}{2} \delta_{mn}$$

- Solution to initial value problem:

$$\sum_{n=1}^N x_n(0) \sin\left(\frac{nk\pi}{N+1}\right) = \frac{N}{2} a_k \cos \delta_k$$

Lumped LC Circuit



$$-L \frac{di_n}{dt} - \frac{1}{C} \int (i_n - i_{n+1}) dt - \frac{1}{C} \int (i_n - i_{n-1}) dt = 0$$

$$\frac{d^2 i_n}{dt^2} + 2\omega_0^2 i_n - \omega_0^2 (i_{n-1} + i_{n+1}) = 0$$

This is the exact same problem as the previous two examples.

Forced Coupled Oscillators

- Qualitative features are the same:
 - Motion can be decoupled into a set of N independent oscillator equations (normal modes)
 - Amplitude of normal mode oscillations are large when driven with the frequency of the normal mode
 - Phase difference approaches $\pi/2$ at resonance
- You should be able to anticipate the qualitative behavior when coupled oscillators are driven by a periodic force.**

Continuous Distributions

Limit as $N \rightarrow \infty$ and $m/\ell \rightarrow \mu$:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

Boundary conditions specified at $x = 0$ and $x = L$:

- Fixed ends: $y(0) = y(L) = 0$
- Maximal motion at ends: $\dot{y}(0) = \dot{y}(L) = 0$
- Mixed boundary conditions

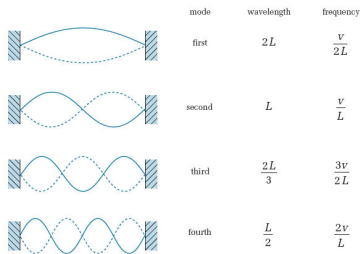
Normal modes will be of the form

$$y_n(x, t) = a_n \sin(k_n x) \cos(\omega_n t - \alpha_n)$$

or $y_n(x, t) = a_n \cos(k_n x) \cos(\omega_n t - \alpha_n)$

Properties of the Solutions

$$y(L, t) \sim \sin k_n L = 0 \Rightarrow k_n L = n\pi$$



$$\lambda_n = \frac{2L}{n}$$

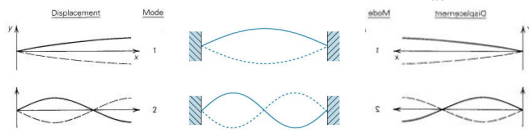
$$\omega_n = \frac{n\pi v}{L}$$

$$f_n = \frac{nv}{2L}$$

Boundary Conditions

Examples:

- String fixed at both ends: $y(0) = y(L) = 0$
- Organ pipe open at one end: $\dot{y}(0) = \dot{y}(L) = 0$
 - Driving end has maximal pressure amplitude
- Organ pipe closed at one end: $\dot{y}(0) = 0, y(L) = 0$
- Transmission line open at one end: $i(L) = 0$
- Transmission line shorted at one end: $v(L) \propto \frac{di(L)}{dt} = 0$



Fourier Analysis

- Normal modes satisfying $y(0) = y(L) = 0$:

$$y_n(x, t) = a_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \alpha_n)$$

- General solution:

$$y(x, t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \alpha_n)$$

- Initial conditions:

$$y(x, 0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos(\alpha_n) = \sum_{n=1}^{\infty} a'_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\dot{y}(x, 0) = -\sum_{n=1}^{\infty} a_n \omega_n \sin\left(\frac{n\pi x}{L}\right) \sin(\alpha_n) = \sum_{n=1}^{\infty} b'_n \sin\left(\frac{n\pi x}{L}\right)$$

Fourier Analysis

- Fourier sine transform:

$$u(x) = \sum_{n=1}^{\infty} a'_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a'_n = \frac{2}{L} \int_0^L u(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

- Fourier cosine transform:

$$b'_n = \frac{2}{L} \int_0^L v(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

Fourier Analysis

$$a'_n = a_n \cos \alpha_n$$

$$b'_n = a_n \omega_n \sin \alpha_n$$

Solve for amplitudes:

$$a_n = \sqrt{a'^2_n + \frac{b'^2_n}{\omega_n^2}}$$

Solve for phase:

$$\tan \alpha_n = \frac{b'_n}{a'_n \omega_n}$$

Fourier Analysis

- Suggestion: don't simply rely on these formulas – use your knowledge of the boundary conditions and initial conditions.**

- Example:

- If you are given $y(x, 0) = 0$ and $y(0) = y(L) = 0$ then you know that solutions are of the form

$$y(x, t) = \sum a_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

- If you are given $y(x, 0) = 0$ and $y'(0) = y'(L) = 0$ then solutions are of the form

$$y(x, t) = \sum_{\text{odd } n} a_n \sin\left(\frac{n\pi x}{L}\right) \sin \omega_n t$$

Progressive Waves

- Far from the boundaries, other descriptions are more transparent:
 $y(x, t) = f(x \pm vt)$

- The Fourier transform gives the frequency components:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cos(kx) dx$$

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \cos(kx) dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(k) \sin(kx) dk$$

$$B(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \sin(kx) dx$$

- Narrow pulse in space \rightarrow wide range of frequencies
- Pulse spread out in space \rightarrow narrow range of frequencies

Properties of Progressive Waves

- Power carried by a wave:
 - String with tension T and mass per unit length μ

$$P = \frac{1}{2} \mu \omega^2 A^2 v = \frac{1}{2} Z \omega^2 A^2$$

- Impedance of the medium:

$$Z = \mu v = T/v$$

- Important properties:
 - Impedance is a property of the medium, not the wave*
 - Energy and power are proportional to the square of the amplitude*

Reflections

- Wave energy is reflected by discontinuities in the impedance of a system
- Reflection and transmission coefficients:
 - The wave is incident and reflected in medium 1
 - The wave is transmitted into medium 2

$$\rho = \frac{Z_2 - Z_1}{Z_1 + Z_2}$$

$$\tau = \frac{2Z_2}{Z_1 + Z_2}$$

Important: when is this negative?

Always positive

- Wave amplitudes:

$$A_r = \rho A_i$$

$$A_t = \tau A_i$$

Reflected and Transmitted Power

- Power is proportional to the square of the amplitude.
 - Reflected power: $P_r = \rho^2 P_i$
 - Transmitted power: $P_t = \tau^2 P_i$
- **You should be able to demonstrate that energy is conserved:**
ie, show that $P_i = P_r + P_t$

That's all for now...

- Study these topics – make sure you understand the examples and assignment questions.
- Midterm exams from previous years are also available on the web.
- Next topics: **waves applied to optics.**
