PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

# Physics 42200

# Waves & Oscillations

Lecture 23 – Review

Spring 2016 Semester

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# Midterm Exam:

| | |
|---|---|
| Date: | Thursday, March 10th |
| Time: | 8:00 – 10:00 pm |
| Room: | MSEE B012 |
| Material: | French, chapters 1-8 |

You can bring one double sided page of notes, formulas, examples, etc.

# Review

1. Simple harmonic motion (one degree of freedom)
   - mass/spring, pendulum, floating objects, RLC circuits
   - damped harmonic motion
2. Forced harmonic oscillators
   - amplitude/phase of steady state oscillations
   - transient phenomena
3. Coupled harmonic oscillators
   - masses/springs, coupled pendula, RLC circuits
   - forced oscillations
4. Uniformly distributed discrete systems
   - masses on string fixed at both ends
   - lots of masses/springs

# Review

5. Continuously distributed systems (standing waves)
   - string fixed at both ends
   - sound waves in pipes (open end/closed end)
   - transmission lines
   - Fourier analysis
6. ~~Progressive waves in continuous systems~~
   - ~~reflection/transmission coefficients~~

# Simple Harmonic Motion

- Any system in which the force is opposite the displacement will oscillate about a point of stable equilibrium
- If the force is proportional to the displacement it will undergo simple harmonic motion
- Examples:
  - Mass/massless spring
  - Elastic rod (characterized by Young's modulus)
  - Floating objects
  - Torsion pendulum (shear modulus)
  - Simple pendulum
  - Physical pendulum
  - LC circuit

# Simple Harmonic Motion

- You should be able to draw a free-body diagram and express the force in terms of the displacement.
- Use Newton's law: $m\ddot{x} = F$ or $I\ddot{\theta} = N$
- Write it in standard form:

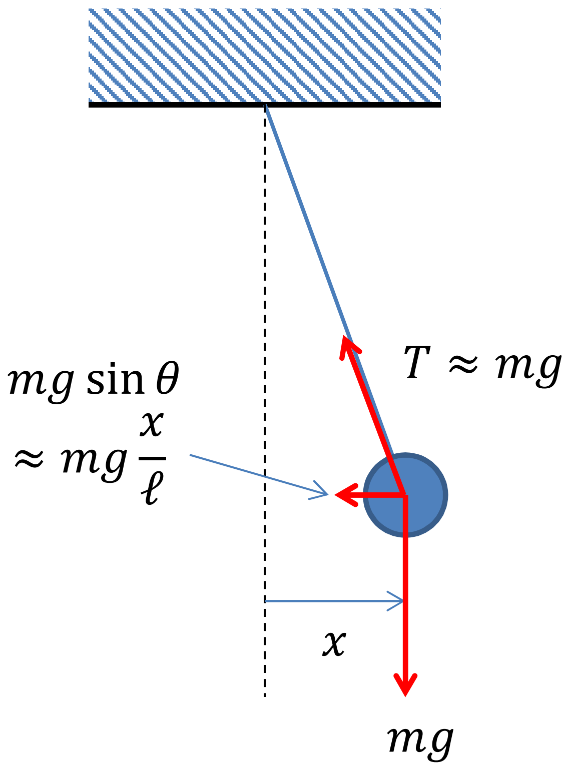
$$\ddot{x} + \omega_0^2 x = 0$$

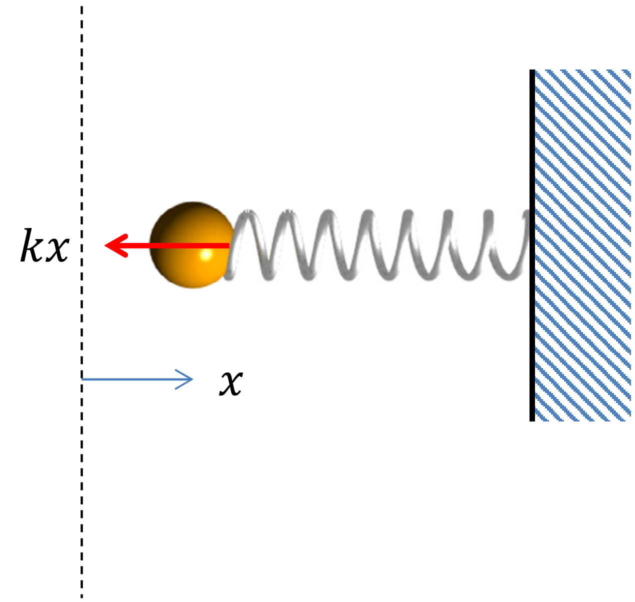
- Solutions are of the form:

$$x(t) = A\cos(\omega_0 t - \delta)$$
$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

- ***You must be able to use the initial conditions to solve for the constants of integration***

# Examples

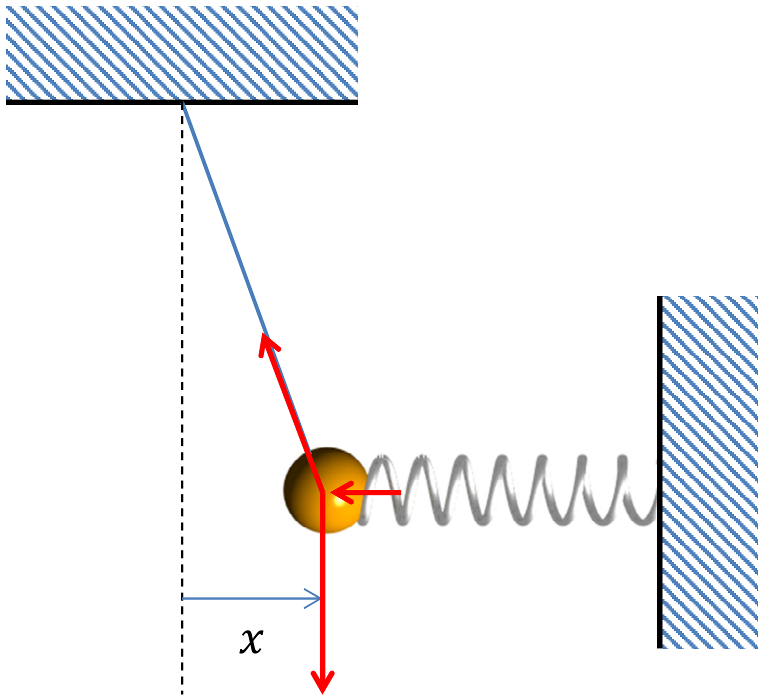



$$m\ddot{x} = -mgx/\ell$$

$$m\ddot{x} = -kx$$

# Examples




$$m\ddot{x} = ?$$

# Damped Harmonic Motion

- Damping forces remove energy from the system
- We will only consider cases where the force is proportional to the velocity: $F = -bv$
- You should be able to construct a free-body diagram and write the resulting equation of motion:

$$m\ddot{x} + b\dot{x} + kx = 0$$

  – You should be able to write it in the standard form:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

- ***You must be able to solve this differential equation!***

# Damped Harmonic Motion

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

Let $x(t) = Ae^{\alpha t}$

- Characteristic polynomial:

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0$$

- Roots (use the quadratic formula):

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}$$

- Classification of solutions:
  - Over-damped: $\gamma^2/4 - (\omega_0)^2 > 0$ (distinct real roots)
  - Critically damped: $\gamma^2/4 = (\omega_0)^2$ (one root)
  - Under-damped: $\gamma^2/4 - (\omega_0)^2 < 0$ (complex roots)

# Damped Harmonic Motion

- Over-damped motion: $\gamma^2/4 - (\omega_0)^2 > 0$

$$x(t) = Ae^{-\frac{\gamma}{2}t}e^{t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}} + Be^{-\frac{\gamma}{2}t}e^{-t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}}$$

- Under-damped motion: $\gamma^2/4 - (\omega_0)^2 < 0$

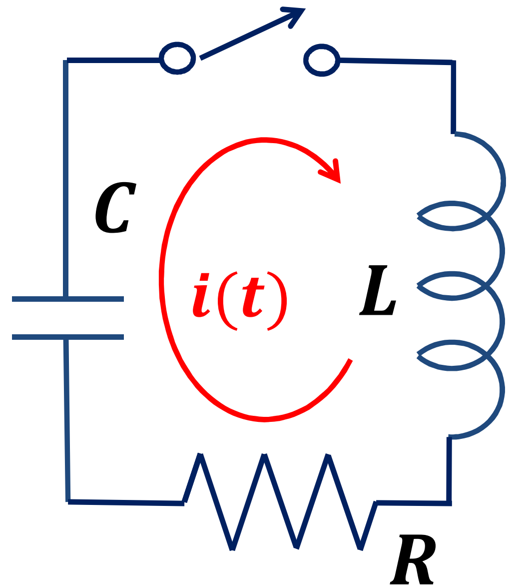
$$x(t) = Ae^{-\frac{\gamma}{2}t}e^{it\sqrt{(\omega_0)^2-\frac{\gamma^2}{4}}} + Be^{-\frac{\gamma}{2}t}e^{-it\sqrt{(\omega_0)^2-\frac{\gamma^2}{4}}}$$

- Critically damped motion:

$$x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$$

- ***You must be able to use the initial conditions to solve for the constants of integration***

# Example




Sum of potential differences:

$$-L\frac{di}{dt} - i(t)R - \frac{1}{C}\left(Q_0 + \int_0^t i(t)dt\right) = 0$$

Initial charge, $Q_0$, defines the initial conditions.

# Example

$$L\frac{di}{dt} + i(t)R + \frac{1}{C}\left(Q_0 + \int_0^t i(t)dt\right) = 0$$

Differentiate once with respect to time:

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = 0$$

$$\frac{d^2 i}{dt^2} + \gamma\frac{di}{dt} + \omega_0^2 i(t) = 0$$

***Remember, the solution is $\boldsymbol{i(t)}$ but the initial conditions might be in terms of $\boldsymbol{Q(t) = Q_0 + \int i(t)dt}$***

(See examples from the lecture notes…)

# Forced Harmonic Motion

- Now the differential equation is
$$m\ddot{x} + b\dot{x} + kx = F(\omega) = F_0 \cos \omega t$$
- Driving function is not always given in terms of a real force… (think about non-inertial reference frames):
$$\ddot{y} + \gamma\dot{y} + \omega_0^2 y = -\frac{d^2\eta}{dt^2} = C\omega^2 \cos \omega t$$
- General properties:
  - Steady state properties: $t \gg 1/\gamma$
  - Solution is $y(t) = A\cos(\omega t - \delta)$
  - Amplitude, $A$, and phase, $\delta$, depend on $\omega$

# Forced Harmonic Motion

"Q" quantifies the amount of damping:

$$Q = \frac{\omega_0}{\gamma}$$

(large Q means small damping force)

$$A(\omega) = \frac{F_0}{k} \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

$$\delta = \tan^{-1}\left(\frac{1/Q}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}}\right)$$

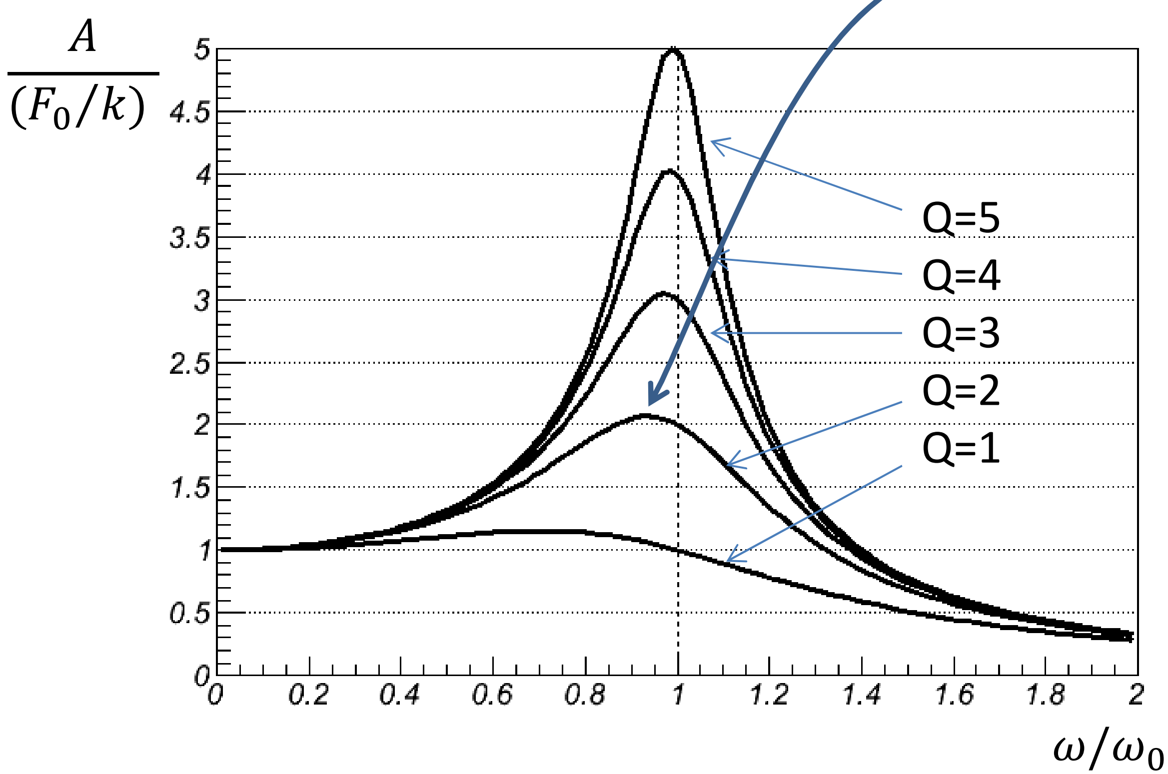
But watch out when $F_0 = C\omega^2$

# Resonance

- Qualitative features: amplitude

$$\frac{\omega_{free}}{\omega_0} = 1$$

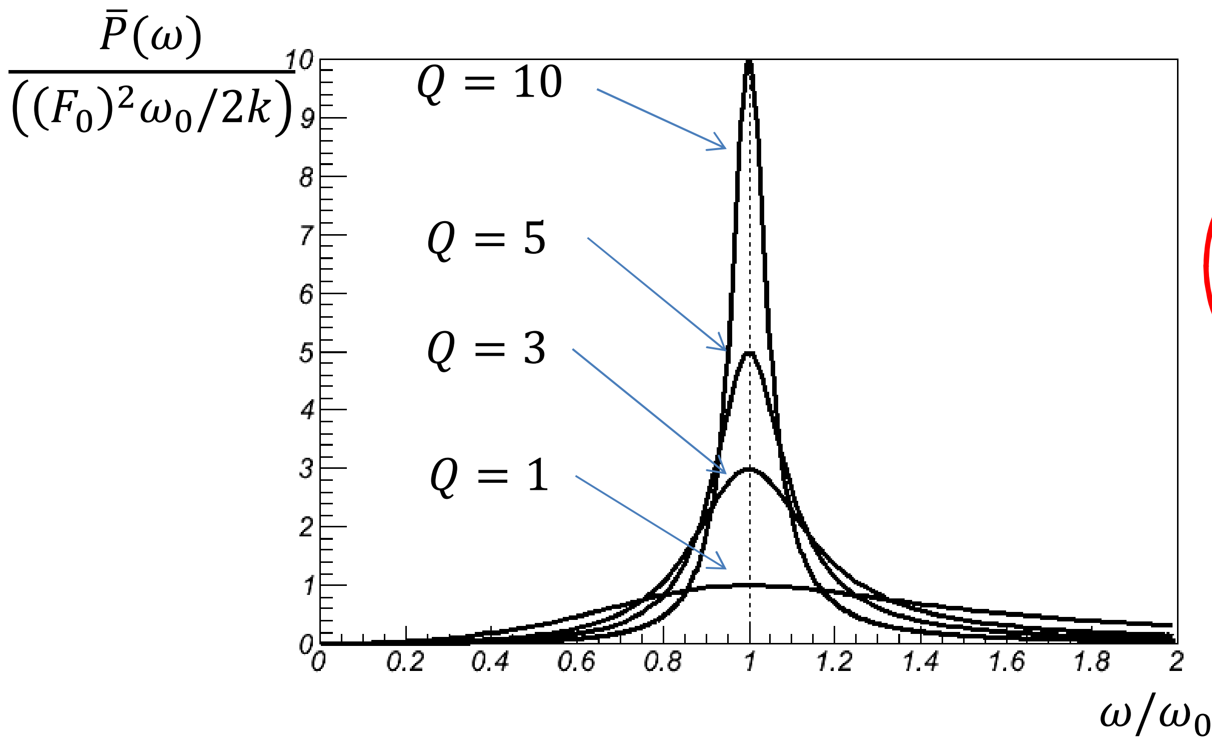
$$\omega_{free} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$



# Average Power

- The rate at which the oscillator absorbs energy is:

$$\bar{P}(\omega) = \frac{(F_0)^2\omega_0}{2kQ}\frac{1}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}$$




Full-Width-at-Half-Max:

$$\boldsymbol{FWHM = \frac{\omega_0}{Q} = \gamma}$$

# Resonance

- Qualitative features: phase shift

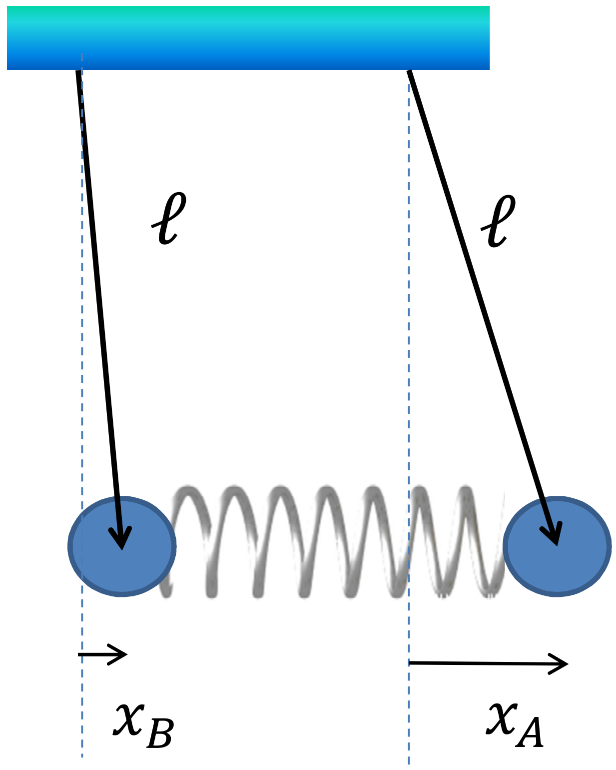
$$\delta = \tan^{-1}\left(\frac{1/Q}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}}\right)$$

$$\delta \to 0 \text{ at low frequencies}$$

$$\delta \to \pi \text{ at high frequencies}$$

$$\boldsymbol{\delta = \frac{\pi}{2}} \textbf{ when } \boldsymbol{\omega = \omega_0}$$

# Coupled Oscillators




- Restoring force on pendulum A:
  $$F_A = -k(x_A - x_B)$$
- Restoring force on pendulum B:
  $$F_B = k(x_A - x_B)$$

$$m\ddot{x}_A + \frac{mg}{\ell}x_A + k(x_A - x_B) = 0$$
$$m\ddot{x}_B + \frac{mg}{\ell}x_B - k(x_A - x_B) = 0$$

# Coupled Oscillators

- ***You must be able to draw the free-body diagram and set up the system of equations.***

$$m\ddot{x}_A + \frac{mg}{\ell}x_A + k(x_A - x_B) = 0$$

$$m\ddot{x}_B + \frac{mg}{\ell}x_B - k(x_A - x_B) = 0$$

- ***You must be able to write this system as a matrix equation.***

$$\begin{pmatrix} \ddot{x}_A \\ \ddot{x}_B \end{pmatrix} + \begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 \end{pmatrix} \begin{pmatrix} x_A(t) \\ x_B(t) \end{pmatrix} = 0$$

# Coupled Oscillators

- Assume solutions are of the form

$$\begin{pmatrix} x_A(t) \\ x_B(t) \end{pmatrix} = \begin{pmatrix} x_A \\ x_B \end{pmatrix} \cos(\omega t - \delta)$$

- Then,

$$\begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 - \omega^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 - \omega^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = 0$$

- ***You must be able to calculate the eigenvalues of a 2x2 or 3x3 matrix.***
    - ***Calculate the determinant***
    - ***Calculate the roots by factoring the determinant or using the quadratic formula.***
- These are the frequencies of the normal modes of oscillation.

# Coupled Oscillators

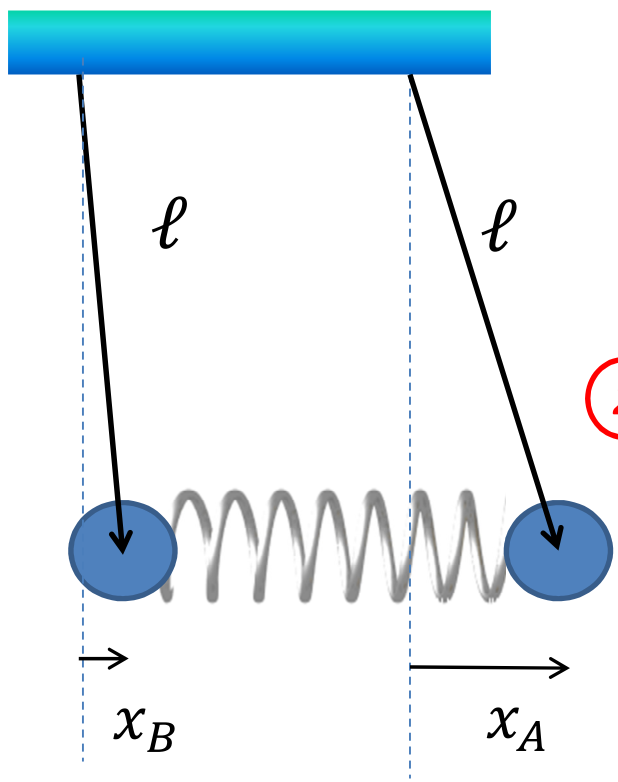
- ***You must be able to calculate the eigenvectors of a 2x2 or 3x3 matrix***
- General solution:

$$\vec{x}(t) = \boldsymbol{A}\vec{x}_1 \cos(\omega_1 t - \boldsymbol{\alpha}) + \boldsymbol{B}\vec{x}_2 \cos(\omega_2 t - \boldsymbol{\beta}) + \cdots$$

- ***You must be able to solve for the constants of integration using the initial conditions.***

# Coupled Discrete Systems

- ***The general method of calculating eigenvalues will always work, but for simple systems you should be able to decouple the equations by a change of variables.***




1
$$\begin{cases} m\ddot{x}_A + \dfrac{mg}{\ell}x_A + k(x_A - x_B) = 0 \\ m\ddot{x}_B + \dfrac{mg}{\ell}x_B - k(x_A - x_B) = 0 \end{cases}$$

2
$$\begin{cases} \ddot{x}_A + [(\omega_0)^2 + (\omega_c)^2]x_A - (\omega_c)^2 x_B = 0 \\ \ddot{x}_B + [(\omega_0)^2 + (\omega_c)^2]x_B - (\omega_c)^2 x_A = 0 \\ \omega_0 = \sqrt{g/\ell},\ \omega_c = \sqrt{k/m} \end{cases}$$

3
$$\begin{cases} q_1 = x_A + x_B \\ q_2 = x_A - x_B \end{cases}$$

4
$$\begin{cases} \ddot{q}_1 + (\omega_0)^2 q_1 = 0 \\ \ddot{q}_2 + (\omega')^2 q_2 = 0 \end{cases}$$

# Forced Oscillations

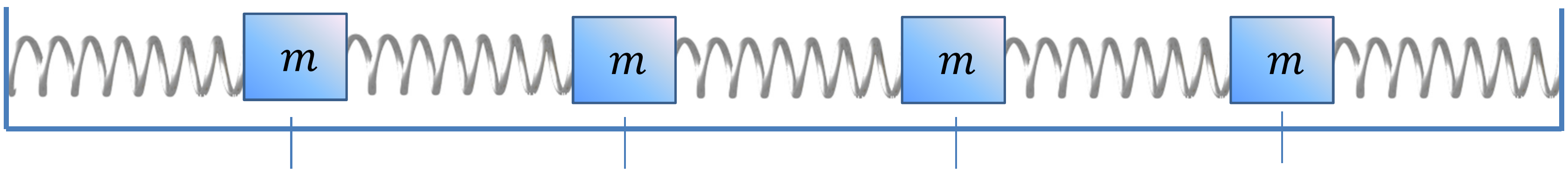
- We mainly considered the qualitative aspects
  - We did not analyze the behavior when damping forces were significant
- Main features:
  - Resonance occurs at each normal mode frequency
  - Phase difference is $\delta = \pi/2$ at resonance
- Example: $x_A$ driven by the force $F(\omega) = F_0\, cos\, \omega t$
  - Calculate force term applied to normal coordinates

$$F_1(\omega) = F_2(\omega) = F_0 \cos \omega t$$

  - Reduced to two one-dimensional forced oscillators:

$$\ddot{q}_1 + (\omega_0)^2 q_1 = F_0/m \cos \omega t$$
$$\ddot{q}_2 + (\omega')^2 q_2 = F_0/m \cos \omega t$$

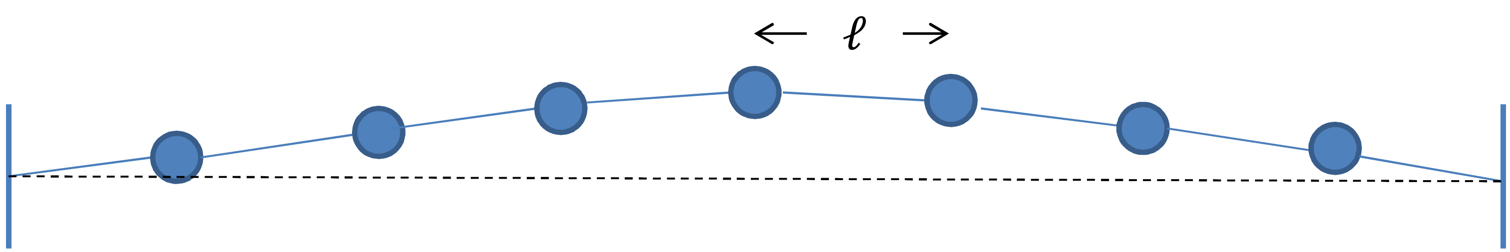
# Uniformly Distributed Discrete Systems




Equations of motion for masses in the middle:

$$\ddot{x}_i + 2(\omega_0)^2 x_i - (\omega_0)^2 (x_{i-1} + x_{i+1}) = 0$$

$$(\omega_0)^2 = k/m$$




$$\ddot{y}_n + 2(\omega_0)^2 y_n - (\omega_0)^2 (y_{n+1} + y_{n-1}) = 0$$

$$(\omega_0)^2 = T/m\ell$$

# Uniformly Distributed Discrete Masses

- Proposed solution:

$$x_n(t) = A_n \cos \omega t$$

$$\frac{A_{n-1} + A_{n+1}}{A_n} = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

- We solved this to determine $A_n$ and $\omega_k$:

$$A_{n,k} = C \sin\left(\frac{nk\pi}{N+1}\right)$$

Amplitude of mass $n$ oscillating in normal mode $k$

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

Frequency of normal mode $k$

- General solution:

$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \delta_k)$$

# Vibrations of Continuous Systems

- General solution for mass $n$:

$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right)\cos(\omega_k t - \delta_k)$$

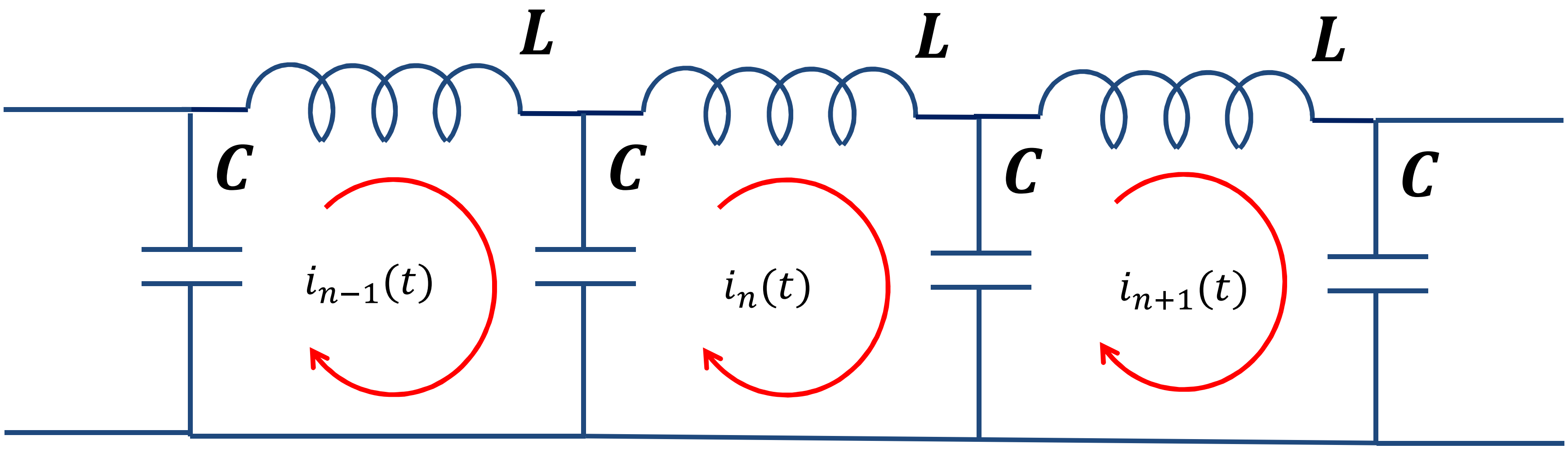
- Orthogonality relation:

$$\sum_{k=1}^{N} \sin\left(\frac{mk\pi}{N+1}\right)\sin\left(\frac{nk\pi}{N+1}\right) = \frac{N}{2}\,\delta_{mn}$$

- Solution to initial value problem:

$$\sum_{n=1}^{N} x_n(0)\sin\left(\frac{nk\pi}{N+1}\right) = \frac{N}{2} a_k\, cos\, \delta_k$$

# Lumped LC Circuit




$$-L\frac{di_n}{dt} - \frac{1}{C}\int (i_n - i_{n+1})dt \; - \frac{1}{C}\int (i_n - i_{n-1})dt = 0$$

$$\frac{d^2 i_n}{dt^2} + 2\omega_0^2 i_n - \omega_0^2(i_{n-1} + i_{n+1}) = 0$$

*This is the exact same problem as the previous two examples.*

# Forced Coupled Oscillators

- Qualitative features are the same:
  - Motion can be decoupled into a set of $N$ independent oscillator equations (normal modes)
  - Amplitude of normal mode oscillations are large when driven with the frequency of the normal mode
  - Phase difference approaches $\pi/2$ at resonance
- ***You should be able to anticipate the qualitative behavior when coupled oscillators are driven by a periodic force.***

# Continuous Distributions

Limit as $N \to \infty$ and $m/\ell \to \mu$:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Boundary conditions specified at $x = 0$ and $x = L$:

- Fixed ends: $y(0) = y(L) = 0$
- Maximal motion at ends: $\dot{y}(0) = \dot{y}(L) = 0$
- Mixed boundary conditions

Normal modes will be of the form

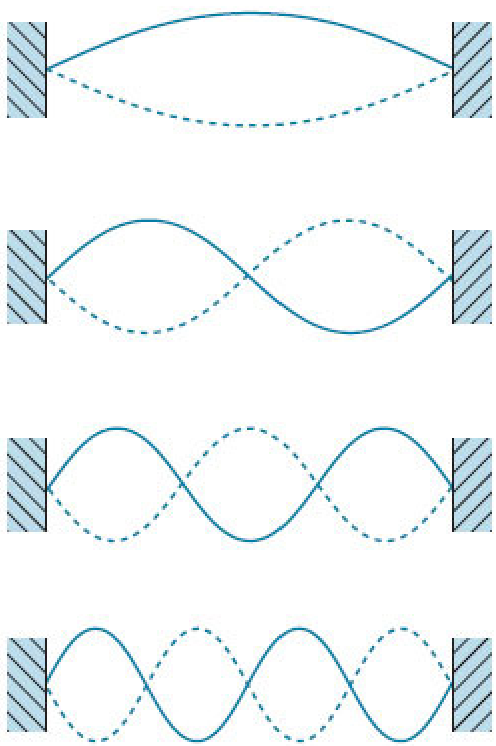
$$y_n(x,t) = a_n \sin(k_n x)\cos(\omega_n t - \alpha_n)$$

or

$$y_n(x,t) = a_n \cos(k_n x)\cos(\omega_n t - \alpha_n)$$

# Properties of the Solutions

$$y(L,t) \sim \sin k_n L = 0 \quad \Rightarrow \quad k_n L = n\pi$$



| mode | wavelength | frequency |
|---|---|---|
| first | $2L$ | $\frac{v}{2L}$ |
| second | $L$ | $\frac{v}{L}$ |
| third | $\frac{2L}{3}$ | $\frac{3v}{2L}$ |
| fourth | $\frac{L}{2}$ | $\frac{2v}{L}$ |

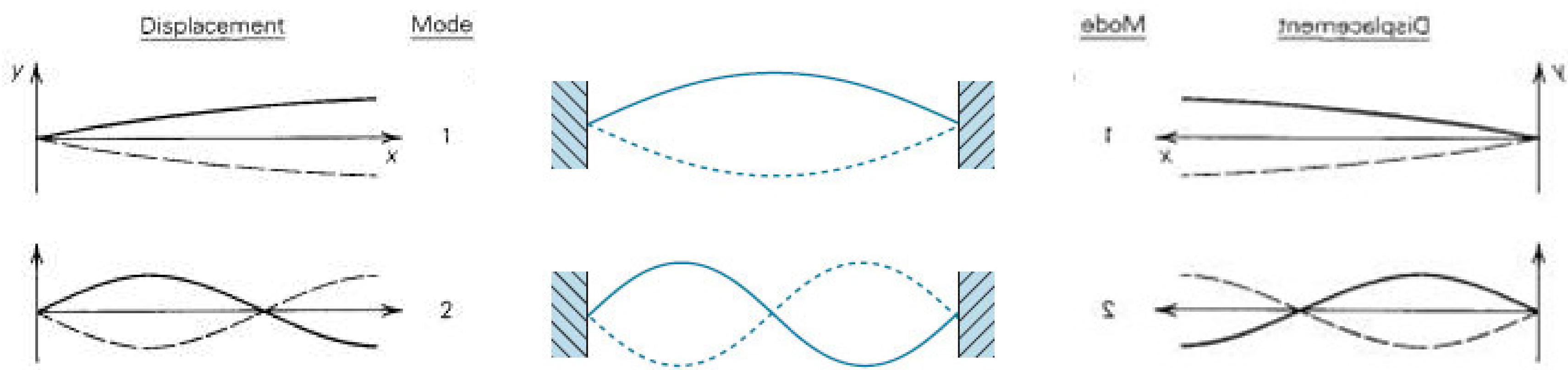
$$\lambda_n = \frac{2L}{n}$$

$$\omega_n = \frac{n\pi v}{L}$$

$$f_n = \frac{nv}{2L}$$

# Boundary Conditions

- Examples:
  - String fixed at both ends: $y(0) = y(L) = 0$
  - Organ pipe open at one end: $\dot{y}(0) = \dot{y}(L) = 0$
    - Driving end has maximal pressure amplitude
  - Organ pipe closed at one end: $\dot{y}(0) = 0, y(L) = 0$
  - Transmission line open at one end: $i(L) = 0$
  - Transmission line shorted at one end: $v(L) \propto \frac{di(L)}{dt} = 0$



# Fourier Analysis

- Normal modes satisfying $y(0) = y(L) = 0$:

$$y_n(x,t) = a_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \alpha_n)$$

- General solution:

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \alpha_n)$$

- Initial conditions:

$$y(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos(\alpha_n) = \sum_{n=1}^{\infty} a'_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\dot{y}(x,0) = -\sum_{n=1}^{\infty} a_n \omega_n \sin\left(\frac{n\pi x}{L}\right)\sin(\alpha_n) = \sum_{n=1}^{\infty} b'_n \sin\left(\frac{n\pi x}{L}\right)$$

# Fourier Analysis

- Fourier sine transform:

$$u(x) = \sum_{n=1}^{\infty} a'_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a'_n = \frac{2}{L}\int_0^L u(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

- Fourier cosine transform:

$$b'_n = \frac{2}{L}\int_0^L v(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

# Fourier Analysis

$$a'_n = a_n \cos \alpha_n$$
$$b'_n = a_n \omega_n \sin \alpha_n$$

Solve for amplitudes:

$$a_n = \sqrt{a'^2_n + \frac{b'^2_n}{\omega_n^2}}$$

Solve for phase:

$$\tan \alpha_n = \frac{b'_n}{a'_n \omega_n}$$

# Fourier Analysis

- ***Suggestion: don't simply rely on these formulas – use your knowledge of the boundary conditions and initial conditions.***
- Example:
  - If you are given $\dot{y}(x,0) = 0$ and $y(0) = y(L) = 0$ then you know that solutions are of the form

$$y(x,t) = \sum a_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

  - If you are given $y(x,0) = 0$ and $y(0) = y(L) = 0$ then solutions are of the form

$$y(x,t) = \sum_{odd\ n} a_n \sin\left(\frac{n\pi x}{L}\right) \sin \omega_n t$$

# Progressive Waves

- Far from the boundaries, other descriptions are more transparent:

$$y(x,t) = f(x \pm vt)$$

- The Fourier transform gives the frequency components:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\cos(kx)\,dx$$

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\cos(kx)\,dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(k)\sin(kx)\,dk\ B(k)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\sin(kx)\,dx$$

- Narrow pulse in space ➔ wide range of frequencies
- Pulse spread out in space ➔ narrow range of frequencies

# Properties of Progressive Waves

- Power carried by a wave:
  - String with tension $T$ and mass per unit length $\mu$

$$P = \frac{1}{2}\mu\omega^2 A^2 v = \frac{1}{2} Z\omega^2 A^2$$

- Impedance of the medium:

$$Z = \mu v = T/v$$

- Important properties:
  - ***Impedance is a property of the medium, not the wave***
  - ***Energy and power are proportional to the square of the amplitude***

# Reflections

- Wave energy is reflected by discontinuities in the impedance of a system
- Reflection and transmission coefficients:
  - The wave is incident and reflected in medium 1
  - The wave is transmitted into medium 2

$$\boldsymbol{\rho} = \frac{\boldsymbol{Z_2 - Z_1}}{\boldsymbol{Z_1 + Z_2}}$$

Important: when is this negative?

$$\boldsymbol{\tau} = \frac{\boldsymbol{2Z_2}}{\boldsymbol{Z_1 + Z_2}}$$

Always positive

- Wave amplitudes:

$$A_r = \rho A_i$$
$$A_t = \tau A_i$$

# Reflected and Transmitted Power

- Power is proportional to the square of the amplitude.
  - Reflected power: $P_r = \rho^2 P_i$
  - Transmitted power: $P_t = \tau^2 P_i$
- ***You should be able to demonstrate that energy is conserved:***

  ***ie, show that*** $\boldsymbol{P_i = P_r + P_t}$

# That’s all for now…

- Study these topics – make sure you understand the examples and assignment questions.
- Midterm exams from previous years are also available on the web.
- Next topics: ***waves applied to optics.***