PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 22 – French, Chapter 8

Spring 2016 Semester

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Midterm Exam:

Date: Thursday, March 10th
Time: 8:00 – 10:00 pm
Room: MSEE B012
Material: French, chapters 1-8

You can bring one double sided page of notes, formulas, examples, etc.

Waves in Two Dimensions

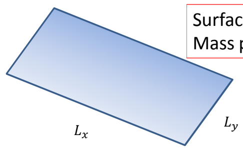
- All systems like this must satisfy the wave equation:

$$\nabla^2 p = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

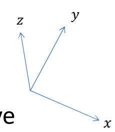
1. Find the kinds of solutions that satisfy the *boundary conditions* (normal modes).
2. Calculate the frequencies of the normal modes.
3. Solve for the constants of integration that satisfy the *initial conditions*.

Waves in Two Dimensions

- Consider a thin rectangular membrane:

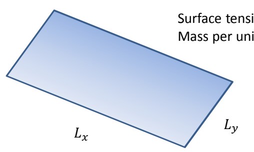


Surface tension, S
Mass per unit area, σ



- We want to find solutions to the wave equation, $z(x, y, t)$.

Waves in Two Dimensions



Surface tension, S
Mass per unit area, σ

- Wave equation:

$$\nabla^2 z = \frac{\sigma}{S}\frac{\partial^2 z}{\partial t^2}$$

- Boundary conditions (in this example):

$$z(0, y, t) = 0 \text{ and } z(L_x, y, t) = 0$$
$$z(x, 0, t) = 0 \text{ and } z(x, L_y, t) = 0$$

Waves in Two Dimensions

$$z(0, y, t) = 0 \text{ and } z(L_x, y, t) = 0$$
$$z(x, 0, t) = 0 \text{ and } z(x, L_y, t) = 0$$

- Proposed solution:

$$z(x, y, t) = C_{n_1 n_2} \sin\left(\frac{n_1 \pi x}{L_x}\right) \sin\left(\frac{n_2 \pi y}{L_y}\right) \cos\left(\omega_{n_1 n_2} t + \varphi_{n_1 n_2}\right)$$

- We might anticipate that $\varphi_{n_1 n_2} = 0$, depending on the initial conditions, or just set it to zero if we are mainly interested in steady state behavior or general properties of the solution.
- In this case:

$$z(x, y, t) = C_{n_1 n_2} \sin\left(\frac{n_1 \pi x}{L_x}\right) \sin\left(\frac{n_2 \pi y}{L_y}\right) \cos\left(\omega_{n_1 n_2} t\right)$$

Waves in Two Dimensions

$$z(x,y,t) = C_{n_1 n_2} \sin\left(\frac{n_1 \pi x}{L_x}\right) \sin\left(\frac{n_2 \pi y}{L_y}\right) \cos\left(\omega_{n_1 n_2} t\right)$$

- Derivatives:

$$\frac{\partial^2 z}{\partial x^2} = -\left(\frac{n_1 \pi}{L_x}\right)^2 z(x,y,t)$$
$$\frac{\partial^2 z}{\partial y^2} = -\left(\frac{n_2 \pi}{L_y}\right)^2 z(x,y,t)$$
$$\frac{\partial^2 z}{\partial t^2} = -\omega_{n_1 n_2}^2 z(x,y,t)$$

Waves in Two Dimensions

- Substitute into the wave equation:

$$\left(\left(\frac{n_1 \pi}{L_x}\right)^2 + \left(\frac{n_2 \pi}{L_y}\right)^2 - \frac{\sigma}{S}\omega_{n_1 n_2}^2\right) z(x,y,t) = 0$$

- Frequencies of normal modes:

$$\omega_{n_1 n_2} = \pm\sqrt{\frac{S}{\sigma}}\left[\left(\frac{n_1 \pi}{L_x}\right)^2 + \left(\frac{n_2 \pi}{L_y}\right)^2\right]^{1/2}$$

Waves in Two Dimensions

- We did the same thing with circular waves:

$$\nabla^2 z = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}$$

- In polar coordinates:

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}$$

- If the waves are rotationally symmetric (they don't have to be) then:

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}$$

Waves in Two Dimensions

- As usual, we can assume that the solution might factor:
  $$z(r,t) = f(r)\cos(\omega t)$$
- Then,
  $$\frac{\partial^2 z}{\partial t^2} = -\omega^2 z(r,t)$$
  $$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{\omega^2}{v^2} z = 0$$
- For convenience, we changed variables: $\rho = kr = r\omega/v$
- As in the rectangular case, we might expect that $v = \sqrt{S/\sigma}$
  $$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \psi(\rho) = 0$$

Waves in Two Dimensions

- We recognized that this was Bessel's equation which has solutions $J_0(kr)$ and $Y_0(kr)$ which can't be written exactly in terms of more familiar analytic functions.
- But, to a very good approximation, we can write:

$$J_0(kr) \approx \sqrt{2/\pi}\frac{\cos(kr - \pi/4)}{\sqrt{kr}}$$
$$Y_0(kr) \approx \sqrt{2/\pi}\frac{\sin(kr - \pi/4)}{\sqrt{kr}}$$
when $kr \gg 1$.

Waves in Two Dimensions

- Boundary conditions: if $\psi(kr) = 0$ when $r = R$ and $\psi(kr)$ remains finite when $r \to 0$ then solutions are of the form $\psi(kr) = J_0(kr)$ and
  $$J_0(kR) \approx \sqrt{2/\pi}\frac{\cos\left(kR - \frac{\pi}{4}\right)}{\sqrt{kR}} = 0$$
- k must satisfy:
  $$kR - \frac{\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, etc \ldots$$
  $$k = \frac{3\pi}{4R}, \frac{7\pi}{4R}, \frac{11\pi}{4R}, etc \ldots$$

Waves in Two Dimensions

- What if the solutions were not rotationally symmetric?
- We could try to look for solutions of the form
$$z(r,\theta,t) = \psi(r)\chi(\theta)\cos(\omega t)$$
- The function $\chi(\theta)$ doesn't really have a boundary, but it must be periodic:
$$\chi(\theta) = \chi(\theta + 2\pi)$$
- A natural choice would be
$$\chi(\theta) = \mathrm{C}\cos m\theta + D\sin m\theta$$
- Then
$$\frac{\partial^2 z}{\partial \theta^2} = m^2 z$$

Waves in Two Dimensions

- When $\chi(\theta) = \mathrm{C}\cos m\theta + D\sin m\theta$, the differential equation becomes:
$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}$$
$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} - \frac{m^2}{r^2}z = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}$$
- Convenient change of variables: $\rho = kr = r\omega/v$
$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \left(\psi(\rho) - \frac{m^2}{\rho^2}\right) = 0$$
- This isn't quite what we had before unless $m = 0$.

Waves in Two Dimensions

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \left(\psi(\rho) - \frac{m^2}{\rho^2}\right) = 0$$

- Now the solutions are the more general Bessel functions: $J_m(kr)$ and $Y_m(kr)$.
- For this course it is sufficient to recognize that these are the solutions… that's all.
- You can look up their properties (eg. roots) or find computer libraries to calculate them if you ever need to.

Example

https://www.youtube.com/watch?v=v4ELxKKT5Rw

Waves in Three Dimensions

- In spherical coordinates (r, θ, ϕ) the Laplacian is:

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) +$$
$$\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

- When $\psi(\vec{r}, t)$ is independent of θ and ϕ then the second line is zero.
- This time, let $\psi(r,t) = \frac{f(r)}{r}\cos\omega t$
- Time derivative: $\frac{\partial^2\psi}{\partial t^2} = -\omega^2\,\psi$

Waves in Three Dimensions

- Let $\psi(r,t) = \frac{f(r)}{r}\cos\omega t$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$$
$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}f(r)\cos\omega t = -\frac{\omega^2}{v^2}\frac{f(r)}{r}\cos\omega t$$
$$\frac{\partial^2 f}{\partial r^2} = -\frac{\omega^2}{v^2}f(r)$$

- We know the solution to this differential equation:

$$f(r) = Ae^{ikr}$$

- The solution to the wave equation is

$$\psi(r,t) = A\frac{e^{ikr}}{r}\cos\omega t$$

Waves in Three Dimensions

- Or we could write
  $$\psi(r,t) = A\frac{\cos k(r \mp vt)}{r}$$
- Waves carry energy proportional to amplitude squared: $\propto 1/r^2$
- The energy is spread out over a surface with area $4\pi r^2$
- Energy is conserved
- Looks like a plane wave at large r