

Physics 42200
Waves & Oscillations

Lecture 22 – French, Chapter 8

Spring 2016 Semester

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Midterm Exam:

Date: Thursday, March 10th

Time: 8:00 – 10:00 pm

Room: MSEE B012

Material: French, chapters 1-8

You can bring one double sided page
of notes, formulas, examples, etc.

Waves in Two Dimensions

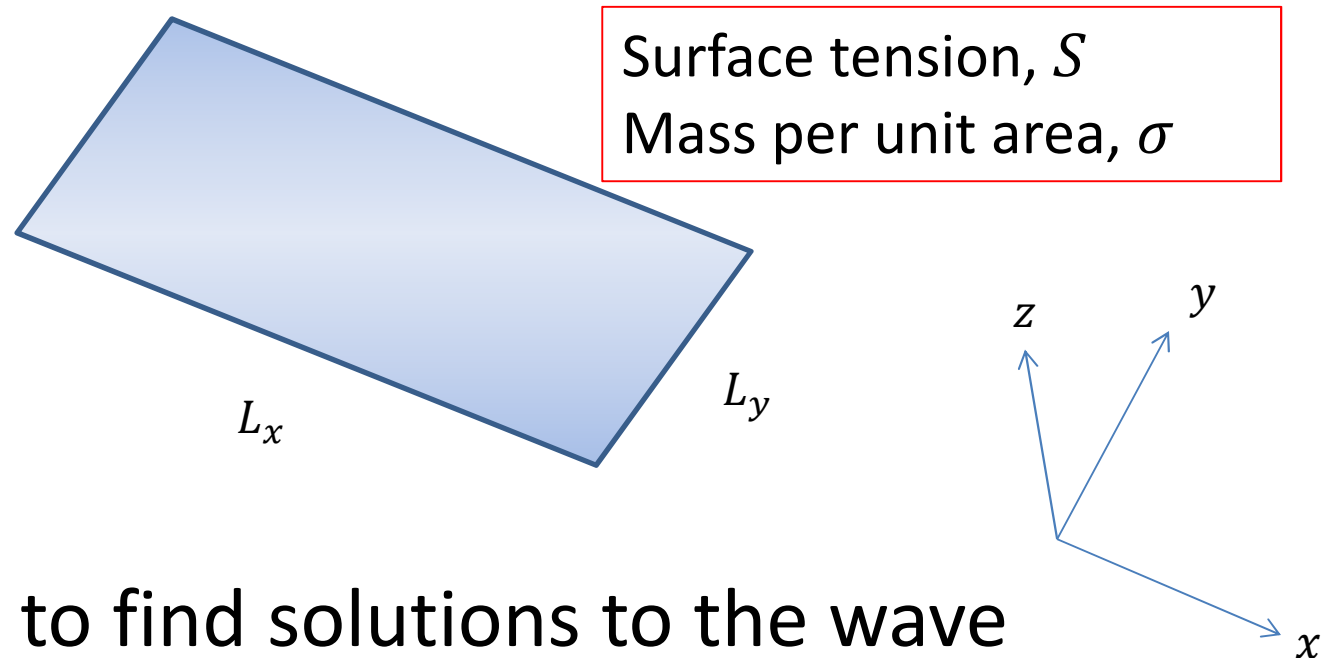
- All systems like this must satisfy the wave equation:

$$\nabla^2 p = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

1. Find the kinds of solutions that satisfy the *boundary conditions* (normal modes).
2. Calculate the frequencies of the normal modes.
3. Solve for the constants of integration that satisfy the *initial conditions*.

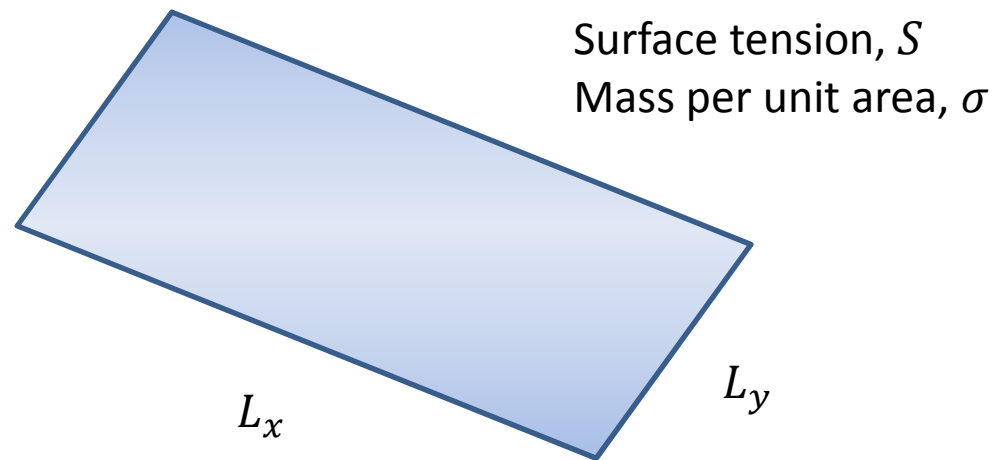
Waves in Two Dimensions

- Consider a thin rectangular membrane:



- We want to find solutions to the wave equation, $z(x, y, t)$.

Waves in Two Dimensions



- Wave equation:

$$\nabla^2 z = \frac{\sigma}{S} \frac{\partial^2 z}{\partial t^2}$$

- Boundary conditions (in this example):

$$z(0, y, t) = 0 \text{ and } z(L_x, y, t) = 0$$

$$z(x, 0, t) = 0 \text{ and } z(x, L_y, t) = 0$$

Waves in Two Dimensions

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- Proposed solution:

$$z(x, y, t) = C_{n_1 n_2} \sin\left(\frac{n_1 \pi x}{L_x}\right) \sin\left(\frac{n_2 \pi y}{L_y}\right) \cos(\omega_{n_1 n_2} t + \varphi_{n_1 n_2})$$

- We might anticipate that $\varphi_{n_1 n_2} = 0$, depending on the initial conditions, or just set it to zero if we are mainly interested in steady state behavior or general properties of the solution.
- In this case:

$$z(x, y, t) = C_{n_1 n_2} \sin\left(\frac{n_1 \pi x}{L_x}\right) \sin\left(\frac{n_2 \pi y}{L_y}\right) \cos(\omega_{n_1 n_2} t)$$

Waves in Two Dimensions

$$z(x, y, t) = C_{n_1 n_2} \sin\left(\frac{n_1 \pi x}{L_x}\right) \sin\left(\frac{n_2 \pi y}{L_y}\right) \cos(\omega_{n_1 n_2} t)$$

- Derivatives:

$$\frac{\partial^2 z}{\partial x^2} = -\left(\frac{n_1 \pi}{L_x}\right)^2 z(x, y, t)$$

$$\frac{\partial^2 z}{\partial y^2} = -\left(\frac{n_2 \pi}{L_y}\right)^2 z(x, y, t)$$

$$\frac{\partial^2 z}{\partial t^2} = -\omega_{n_1 n_2}^2 z(x, y, t)$$

Waves in Two Dimensions

- Substitute into the wave equation:

$$\left(\left(\frac{n_1 \pi}{L_x} \right)^2 + \left(\frac{n_2 \pi}{L_y} \right)^2 - \frac{\sigma}{S} \omega_{n_1 n_2}^2 \right) z(x, y, t) = 0$$

- Frequencies of normal modes:

$$\omega_{n_1 n_2} = \pm \sqrt{\frac{S}{\sigma} \left[\left(\frac{n_1 \pi}{L_x} \right)^2 + \left(\frac{n_2 \pi}{L_y} \right)^2 \right]^{1/2}}$$

Waves in Two Dimensions

- We did the same thing with circular waves:

$$\nabla^2 z = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2}$$

- In polar coordinates:

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2}$$

- If the waves are rotationally symmetric (they don't have to be) then:

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2}$$

Waves in Two Dimensions

- As usual, we can assume that the solution might factor:

$$z(r, t) = f(r) \cos(\omega t)$$

- Then,

$$\frac{\partial^2 z}{\partial t^2} = -\omega^2 z(r, t)$$

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} + \frac{\omega^2}{v^2} z = 0$$

- For convenience, we changed variables: $\rho = kr = r\omega/v$

- As in the rectangular case, we might expect that

$$v = \sqrt{S/\sigma}$$

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \psi(\rho) = 0$$

Waves in Two Dimensions

- We recognized that this was Bessel's equation which has solutions $J_0(kr)$ and $Y_0(kr)$ which can't be written exactly in terms of more familiar analytic functions.
- But, to a very good approximation, we can write:

$$J_0(kr) \approx \sqrt{2/\pi} \frac{\cos(kr - \pi/4)}{\sqrt{kr}}$$
$$Y_0(kr) \approx \sqrt{2/\pi} \frac{\sin(kr - \pi/4)}{\sqrt{kr}}$$

when $kr \gg 1$.

Waves in Two Dimensions

- Boundary conditions: if $\psi(kr) = 0$ when $r = R$ and $\psi(kr)$ remains finite when $r \rightarrow 0$ then solutions are of the form $\psi(kr) = J_0(kr)$ and

$$J_0(kR) \approx \sqrt{2/\pi} \frac{\cos\left(kR - \frac{\pi}{4}\right)}{\sqrt{kR}} = 0$$

- k must satisfy:

$$kR - \frac{\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, etc ...$$
$$k = \frac{3\pi}{4R}, \frac{7\pi}{4R}, \frac{11\pi}{4R}, etc ...$$

Waves in Two Dimensions

- What if the solutions were not rotationally symmetric?

- We could try to look for solutions of the form

$$z(r, \theta, t) = \psi(r)\chi(\theta) \cos(\omega t)$$

- The function $\chi(\theta)$ doesn't really have a boundary, but it must be periodic:

$$\chi(\theta) = \chi(\theta + 2\pi)$$

- A natural choice would be

$$\chi(\theta) = C \cos m\theta + D \sin m\theta$$

- Then

$$\frac{\partial^2 z}{\partial \theta^2} = -m^2 z$$

Waves in Two Dimensions

- When $\chi(\theta) = C \cos m\theta + D \sin m\theta$, the differential equation becomes:

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2}$$
$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} - \frac{m^2}{r^2} z = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2}$$

- Convenient change of variables: $\rho = kr = r\omega/v$

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \left(\psi(\rho) - \frac{m^2}{\rho^2} \right) = 0$$

- This isn't quite what we had before unless $m = 0$.

Waves in Two Dimensions

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \left(\psi(\rho) - \frac{m^2}{\rho^2} \right) = 0$$

- Now the solutions are the more general Bessel functions: $J_m(kr)$ and $Y_m(kr)$.
- For this course it is sufficient to recognize that these are the solutions... that's all.
- You can look up their properties (eg. roots) or find computer libraries to calculate them if you ever need to.

Example

<https://www.youtube.com/watch?v=v4ELxKKT5Rw>

Waves in Three Dimensions

- In spherical coordinates (r, θ, ϕ) the Laplacian is:

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}$$

- When $\psi(\vec{r}, t)$ is independent of θ and ϕ then the second line is zero.
- This time, let $\psi(r, t) = \frac{f(r)}{r} \cos \omega t$
- Time derivative: $\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi$

Waves in Three Dimensions

- Let $\psi(r, t) = \frac{f(r)}{r} \cos \omega t$

$$\begin{aligned}\nabla^2 \psi &= \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi) \\ &= \frac{1}{r} \frac{\partial^2}{\partial r^2} f(r) \cos \omega t = -\frac{\omega^2}{v^2} \frac{f(r)}{r} \cos \omega t \\ \frac{\partial^2 f}{\partial r^2} &= -\frac{\omega^2}{v^2} f(r)\end{aligned}$$

- We know the solution to this differential equation:

$$f(r) = Ae^{ikr}$$

- The solution to the wave equation is

$$\psi(r, t) = A \frac{e^{ikr}}{r} \cos \omega t$$

Waves in Three Dimensions

- Or we could write

$$\psi(r, t) = A \frac{\cos k(r \mp vt)}{r}$$

- Waves carry energy proportional to amplitude squared: $\propto 1/r^2$
- The energy is spread out over a surface with area $4\pi r^2$
- Energy is conserved
- Looks like a plane wave at large r

